Biderivations of complete Lie algebras

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The main result of the paper [1] describes all complete Lie algebras with trivial center and internal derivations, i.e., a Lie algebra with a trivial center and with inner derivations. In particular, semi-simple Lie algebras fall into this category. The work also extends a well-known result on simple Lie algebras obtained by X. Tang in 2018 ([2]). Furthermore, the paper presents some results on symmetric and antisymmetric biderivations.

Proposition 1.1. *Let L be a complete Lie algebra over a field $\mathbb{F}$. B is a biderivation of L if and only if there exist two linear maps $\varphi, \psi \in \operatorname{End}(L)$ such that, for every $x, y \in L$,*

$$B(x,y) = [\varphi(x), y] = [x, \psi(y)] .$$

Theorem 1.2. *Let $L = L_1 \oplus \cdots \oplus L_t$ be a complex semisimple Lie algebra of dimension n, where L_i is a complex simple Lie algebra with $\dim_{\mathbb{C}} L_i = n_i$ for each $i = 1, \ldots, t$, and $n_1 + \cdots + n_t = n$. A bilinear map $B \colon L \times L \to L$ is a biderivation of L if and only if there exist $\lambda_1, \ldots, \lambda_t \in \mathbb{C}$ such that*

$$B(x_1 + \cdots + x_t, y_1 + \cdots + y_t) = \lambda_1 [x_1, y_1] + \cdots + \lambda_t [x_t, y_t] ,$$

with $x_i, y_i \in L_i$, $i = 1, \ldots, t$.

It is worth noting that a linear map $f : L \to L$ is called *commutative* if $[x, f(x)] = 0$ for every $x \in L$. If $\operatorname{char}(\mathbb{F}) \neq 2$, then a linear map $f \colon L \to L$ is commutative if and only if $[f(x), y] = [x, f(y)]$ for every $x, y \in L$. Similarly, f is called *antisymmetric* if and only if $[f(x), y] = -[x, f(y)]$ for all $x, y \in L$. To be more precise, the definition of commutative linear maps can be extended to a broader class of algebraic structures, such as rings.

Corollary 1.3. *Let L be a complete Lie algebra. $B \colon L \to L$ is a symmetric biderivation of L if and only if there exists a unique antisymmetric linear map $\varphi \in \operatorname{End}(L)$ such that $B(x, y) = [\varphi(x), y]$, for every $x, y \in L$.*

Corollary 1.4. *Let L be a complete Lie algebra. $B \colon L \to L$ is an antisymmetric biderivation of L if and only if there exists a unique commutative linear map $\varphi \in \operatorname{End}(L)$ such that $B(x, y) = [\varphi(x), y]$, for every $x, y \in L$.*

References

[1] A. Di Bartolo and G. La Rosa, *Biderivations of complete Lie algebras*, Journal of Algebra and Its Applications, (2023), DOI: `10.1142/S0219498825500161`

[2] X. Tang, *Biderivations of finite-dimensional complex simple Lie algebras*, Linear and Multilinear Algebra, **66**(2), 250-259 (2018), DOI: `10.1080/03081087.2017.1295433`.