

# NATURAL BOUNDARIES OF DIRICHLET SERIES

GAUTAMI BHOWMIK AND JAN-CHRISTOPH SCHLAGE-PUCHTA

ABSTRACT. We prove some conditions on the existence of natural boundaries of Dirichlet series. We show that generically the presumed boundary is the natural one. We also give an application of natural boundaries in determining asymptotic results.

## 1. INTRODUCTION

It is very difficult to say much about the meromorphic continuation of Euler products of Dirichlet series beyond the region of convergence. The only general method to show the existence of a natural boundary is to prove that every point of the presumed boundary is the limit point of either poles or zeros of the function in question. In general, it does not suffice to prove that each point is a limit point of poles or zeros of the single factors, since poles and zeros might cancel.

There are, of course, many examples of special cases where precise information was obtained, as was done by Estermann [2] who proved that if there is an Euler product  $D(s) = \prod_p h(p^{-s})$  where  $h(Y)$  is a ganzwertige polynomial, then  $D(s)$  is absolutely convergent for  $\Re(s) > 1$  and can be meromorphically continued to the half plane  $\Re(s) > 0$ . If  $h(Y)$  is a product of cyclotomic polynomials, then and only then can  $D(s)$  be continued to the whole complex plane. Dahlquist [3] generalised this result to  $h$  being any analytic function with isolated singularities within the unit circle. This line of investigation was continued to several variables. Analytic continuations of multiple zeta functions received a lot of attention in recent years, especially by the Japanese school. The known examples confirm the belief that if there is an obvious candidate for a boundary, it is the natural boundary.

However, in cases like  $D(s) = \prod_p h(p, p^{-s})$ , which occur in the study of algebraic groups (see, for example, [5]), the above belief is yet to be confirmed. Thus a simple case like  $D(s) = \prod_p (1 + p^{-s} + p^{1-2s})$  [6] remains resistant to our understanding. In this paper, we find some conditions such that too much cancellation among potential singularities becomes impossible and a natural boundary is obtained (Theorem 1). This helps us give partial answers to series like the one just cited (Corollary 1).

Our belief in the possibility of meromorphic continuation upto an expected domain is strengthened by a generic consideration. Several authors (see, for example, [8] or [9]) studied Dirichlet series with random coefficients and showed that such series almost surely have natural boundaries. In this paper, the definition of a random series is adapted to serve our interest and we prove that almost surely the series thus defined has meromorphic continuation upto the presumed half-plane (Theorem 3).

Finally, we show that the existence of a natural boundary can help in obtaining  $\Omega$ -results for Dirichlet series associated to counting functions. We prove that if

$D(s) = \sum a(n)n^{-s}$  has a natural boundary at  $\Re s = \sigma$ , then there does not exist an explicit formula of the form  $A(x) := \sum_{n \leq x} a_n = \sum_{\rho} c_{\rho} x^{\rho} + \mathcal{O}(x^{\sigma})$ , where  $\rho$  is a zero of the Riemann zeta function and hence it is possible to obtain a term  $\Omega(x^{\sigma-\epsilon})$  in the asymptotic expression for  $A(x)$ . We treat the Igusa-Weil function for algebraic groups for rank 3 (interpreted as a counting function) as an example of this manoeuvre.

## 2. CRITERIA FOR A BOUNDARY

Combinatorics of sets of integers and real numbers are often an ingredient of the proofs of natural boundary; confer, for instance, Dahlquist's concept of vertex numbers [3]. The following Lemma shows that in an appropriate setting, there cannot be too much cancellations among potential singularities. For a discrete set  $\mathcal{X} \subset [0, \infty)$ , denote by  $\mathcal{X}(x)$  the number of elements of  $\mathcal{X} \cap [0, x]$ .

**Lemma 1.** *Let  $0 < \epsilon < 1$  be given. Let  $\mathcal{P}$  be a set of primes with  $\mathcal{P}((1 + \epsilon)x) - \mathcal{P}(x) \gg x^{\theta} \log^2 x$  and let  $\mathcal{X} \subset [0, T]$  be a discrete set of real numbers satisfying  $\mathcal{X}(x) \ll x \log x$  and  $T \in \mathcal{X}$ . Assume that for all  $p \in \mathcal{P}$  and all  $x \in \mathcal{X}$  with  $\frac{x}{p} \in [1, 1 + \epsilon]$ , there exists some  $n \in \mathbb{N} \setminus \mathcal{P}$  and some  $y \in \mathcal{X}$  with  $\frac{x}{p} = \frac{y}{n}$ . Then we have  $\theta < \frac{\sqrt{5}-1}{2} = 0.618\dots$*

Of course, the condition on the growth of  $\mathcal{X}$  is somewhat arbitrary; the formulation chosen here being dictated by the application, where  $\mathcal{X}$  will be chosen as a set of abscissae of certain zeros of  $\zeta$ .

*Proof.* For each  $p \in \mathcal{P} \cap [\frac{T}{1+\epsilon}, T]$ , there exists some  $y_p \in \mathcal{X}$  such that there is some  $n \notin \mathcal{P}$  with  $\frac{T}{p} = \frac{y_p}{n}$ . For each such  $p$ , choose an integer  $a_p$  with the property that  $a_p y_p \in \mathcal{X}$ , but  $ka_p y_p \notin \mathcal{X}$  for all integers  $k \geq 2$ . Next, for each  $p$  choose prime numbers  $p' \in \mathcal{X} \cap [\frac{a_p y_p}{1+\epsilon}, a_p y_p]$  with  $p' \nmid a_p$ . For each such choice, there exists an element  $z_{p,p'} \in \mathcal{X}$  and an integer  $n' \notin \mathcal{P}$ , such that  $\frac{a_p y_p}{p'} = \frac{z_{p,p'}}{n'}$ . We claim that  $z_{p,p'} = z_{\tilde{p}, \tilde{p}'}$  implies that  $\{p, p'\} = \{\tilde{p}, \tilde{p}'\}$ . In fact, we have

$$z_{p,p'} = z_{\tilde{p}, \tilde{p}'} \Leftrightarrow T \frac{a_p n n'}{p p'} = T \frac{a_{\tilde{p}} \tilde{n} \tilde{n}'}{\tilde{p} \tilde{p}'}$$

By construction, all the integers  $a_p, a_{p'}, n, \tilde{n}, n', \tilde{n}'$  are at most  $T$ , while  $p$  and  $\tilde{p}$  are at least  $\frac{T}{1+\epsilon}$ . Hence, neither  $p$  nor  $\tilde{p}$  cancel, and we either obtain  $p = \tilde{p}$ , or  $p = \tilde{p}'$  and  $p' = \tilde{p}$ . In the latter case the sets  $\{p, p'\}$  and  $\{\tilde{p}, \tilde{p}'\}$  coincide, and we are done. To deal with the first possibility, note that the choice of  $a_p$  and  $n$  depend only on  $p$ , hence,  $z_{p,p'} = z_{\tilde{p}, \tilde{p}'}$  implies  $\frac{n'}{p'} = \frac{\tilde{n}'}{\tilde{p}'}$ . If  $p'$  does not divide  $n'$ , this implies  $p' = \tilde{p}'$ , and we obtain  $\{p, p'\} = \{\tilde{p}, \tilde{p}'\}$  as in the first case. Finally, we show that by construction of  $p'$  and  $n'$ ,  $p'$  can never divide  $n'$ . In fact,  $p' \neq n'$ , since otherwise  $n'$  would be in  $\mathcal{P}$ , contrary to our choice of  $n'$ . Thus,  $\frac{n'}{p'} = k$  would be an integer  $\geq 2$ , and we would obtain  $z_{p,p'} = ka_p y_p$ , which would contradict our definition of  $a_p$ . Hence, we have shown that  $z_{p,p'}$  indeed determines the set  $\{p, p'\}$ . Next, we estimate the number of sets  $\{p, p'\}$  in the above manner. By assumption, there are  $\gg T^{\theta} \log^2 T$  choices for  $p$ . The growth condition for  $\mathcal{X}$  implies that there are at least  $T^{\theta}$  prime numbers  $p$ , such that  $y_p > T^{\theta}$ . For each such prime  $p$ , the number of choices for  $p'$  is  $\gg y_p^{\theta} \log^2 y_p \gg T^{\theta^2} \log^2 T$ , hence, the total number of pairs  $(p, p')$  is

of order of magnitude  $T^{\theta+\theta^2} \log^2 T$ , and the number of unordered sets differs from this quantity by a factor of 2 at most. Hence, we obtain the estimates

$$T^{\theta+\theta^2} \log^2 T \ll |\{z_{p,p'}\}| \leq |\mathcal{X}| \ll T \log T,$$

which implies  $\theta + \theta^2 < 1$ , that is,  $\theta < \frac{\sqrt{5}-1}{2}$ .  $\square$

**Theorem 1.** *Assume the Riemann  $\zeta$ -function has infinitely many zeros off the line  $\frac{1}{2} + it$ . Suppose that  $f$  is a function of the form  $f(s) = \prod_{\nu \geq 1} \zeta(\nu(s - \frac{1}{2}) + \frac{1}{2})^{n_\nu}$  where the exponents  $n_\nu$  are rational integers and the series  $\sum \frac{n_\nu}{2^{\epsilon\nu}}$  converges absolutely for every  $\epsilon > 0$ . Then  $f$  is holomorphic in the half plane  $\Re s > 1$  and has meromorphic continuation in the half plane  $\Re s > \frac{1}{2}$ . Denote by  $\mathcal{P}$  the set of prime numbers  $p$ , such that  $n_p > 0$ , and suppose that for all  $\epsilon > 0$  we have  $\mathcal{P}((1 + \epsilon)x) - \mathcal{P}(x) \gg x^{\frac{\sqrt{5}-1}{2}} \log^2 x$ . Then the line  $\Im s = \frac{1}{2}$  is the natural boundary of  $f$ ; more precisely, every point of this line is accumulation point of zeros of  $f$ .*

*Proof.* Let  $\epsilon > 0$  be given. Then only finitely many factors in the infinite product have a pole in the half-plane  $\Re s > \frac{1}{2} + \epsilon$ , and we have  $\zeta(\nu(s - \frac{1}{2}) + \frac{1}{2}) - 1 \sim 2^{\nu(s - \frac{1}{2}) + \frac{1}{2}}$  for  $\nu \rightarrow \infty$ , thus apart from a discrete subset we have

$$\begin{aligned} f(s) &= \exp\left(\sum_{\nu} n_\nu \log \zeta\left(\nu\left(s - \frac{1}{2}\right) + \frac{1}{2}\right)\right) \\ &= \exp\left(\sum_{\nu} \frac{n_\nu}{2^{\nu(s - \frac{1}{2}) + \frac{1}{2}}} + \mathcal{O}(1)\right), \end{aligned}$$

and by assumption this sum converges absolutely for all  $s$  with  $\Re s > \frac{1}{2}$  this sum converges absolutely for all  $s$  with  $\Re s > \frac{1}{2}$ , hence, apart from a discrete set of poles,  $f$  can be holomorphically continued to the half-plane  $\Re s > \frac{1}{2}$ . We shall now prove that every point of the line  $1/2 + it$  is an accumulation point of zeros or poles of  $f$ . To do so, note first that every point on this line is accumulation point of zeros with real part  $> 1/2$  of factors in the infinite product defining  $f$ . In fact, by assumption there are infinitely many zeros of  $\zeta$  to the right of the line  $\Re s = 1/2$ , thus, for every  $\epsilon > 0$  and every  $t$  there is a zero  $\rho = \sigma + iT$  of  $\zeta$ , such that  $\mathcal{P}(T/t) - \mathcal{P}(T/((1 + \epsilon)t)) \gg (T/t)^\theta \log^2(T/t)$ , where  $\theta = \frac{\sqrt{5}-1}{2}$ . In particular, there exists a prime number  $p$  with  $n_p > 0$ , such that  $T/p \in [t, (1 + \epsilon)t]$ . Hence, to prove our claim, we have to show that this zero cannot be cancelled by poles stemming from other factors. We cannot do so for a single point, however, using Lemma 1, we can show that not all such poles or zeros can be cancelled. In fact, let  $\mathcal{X}$  be the set imaginary parts of zeros of  $\zeta$  lying on the line passing through  $\frac{1}{2}$  and  $\rho$  and having positive imaginary part. Let  $T$  be the maximum of  $\mathcal{X}$ , that is  $\mathcal{X} \subset [0, T]$ . Since the number of all zeros of  $\zeta$  with imaginary part  $\leq x$  is of magnitude  $\mathcal{O}(x \log x)$ , we have a fortiori  $\mathcal{X}(x) \ll (x \log x)$ . If  $\frac{\rho-1/2}{p} + \frac{1}{2}$  is not a zero of  $f$ , there has to be some integer  $\nu$  and a zero  $\rho'$  of  $\zeta$ , such that  $n_\nu < 0$ , and  $\frac{\rho-1/2}{p} = \frac{\rho'-1/2}{\nu}$ , that is,  $\rho'$  is on the line through  $\frac{1}{2}$  and  $\rho$ , and has positive imaginary part, thus,  $\Im \rho' \in \mathcal{X}$ . Moreover, for every  $p \in \mathcal{P}$  we have  $n_p > 0$ , whereas  $n_\nu < 0$ , thus,  $\nu \notin \mathcal{P}$ . Since we are not restricted in our choice of  $p$  and  $\rho$  except for the conditions  $p \in \mathcal{P}$  and  $\frac{\Im \rho}{p} \in [t, (1 + \epsilon)t]$ , we find that we can apply Lemma 1 to deduce  $\theta < \frac{\sqrt{5}-1}{2}$ . However, this contradicts our assumption on the density of  $\mathcal{P}$ , which show that there is some  $p \in \mathcal{P}$  and a zero  $\rho$  of  $\zeta$ , such that  $\frac{\rho-1/2}{p} + \frac{1}{2}$  is a zero

of  $f$ , that is, in every square of the form  $\{s : \Re s \in [\frac{1}{2}, \frac{1}{2} + \epsilon], \Im s \in [t, t + \epsilon]\}$ , there is a zero of  $f$ , that is, every point of the line  $\Re s = \frac{1}{2}$  is accumulation point of zeros of  $f$ , and since  $f$  is not identically zero, this line forms a natural boundary.  $\square$

We can use Theorem 1 to give a partial solution to an “embarrassingly innocuous looking” case (see [6, sec. 3.2.4]).

**Corollary 1.** *Suppose that there are infinitely many zeros of  $\zeta$  off the line  $\frac{1}{2} + it$ . Then the function*

$$f(s) = \prod_p \left(1 + p^{-s} + p^{1-2s}\right)$$

has meromorphic continuation to the half plane  $\Re s > \frac{1}{2}$ , and the line  $\Re s = \frac{1}{2}$  is the natural boundary of  $f$ .

*Proof.* The function  $f$  can be expanded into a product of  $\zeta$ -functions as follows:

$$\begin{aligned} f(s) &= \frac{\zeta(s)\zeta(2s-1)\zeta(3s-1)}{\zeta(2s)\zeta(4s-2)} R(s) \\ &\times \prod_{m \geq 1} \frac{\zeta((4m+1)s-2m)}{\zeta((4m+3)s-2m-1)\zeta((8m+2)s-4m)}, \end{aligned}$$

where  $R(s)$  is a function holomorphic in some half-plane strictly larger than the half-plane  $\Re s > \frac{1}{2}$ . Denote by  $D$  the infinite product on the right of the last equation. Then we have

$$D(s) = \prod_{m \geq 1} \frac{\zeta((4m+1)s-2m)}{\zeta((4m+3)s-2m-1)} \prod_{m \geq 1} \zeta((8m+2)s-4m)^{-1} = \prod_1 \times \prod_2,$$

say.  $\prod_1$  is of the form considered in Theorem 1, whereas for  $\Re s > \frac{1}{2}$ ,  $\prod_2$  is an absolutely convergent product of values of  $\zeta$  in the half-plane  $\Re s > 1$ , thus,  $\prod_2$  is holomorphic and non-vanishing in the half-plane  $\Re s > \frac{1}{2}$ , and therefore cannot interfere with zeros of  $\prod_1$ . Hence, every point of the line  $\Re s = \frac{1}{2}$  is an accumulation point of zeros of  $D$ , and  $D$  cannot be continued meromorphically beyond this line.  $\square$

Another application is the following, which partially resolves a class of polynomials considered in [6, Theorem 3.23].

**Theorem 2.** *Let  $D(s) = \prod W(p, p^{-s}) = \prod \zeta(ms - n)^{c_{nm}}$  be a Dirichlet-series, such that all local zeros are to the left of the line  $\Re s = \beta$ , where  $\beta$  is the largest limit point of the set  $\{\frac{n}{m} : c_{nm} \neq 0\}$ . Suppose that the number  $P(x)$  of prime numbers  $p$  such that there is some  $n$  with  $c_{np} \neq 0$  and  $n/p + 1/2p > \beta$  satisfies  $P((1 + \epsilon)x) - P(x) \gg x^{\frac{\sqrt{5}-1}{2}} \log^2 x$ . Then  $\beta$  is the natural boundary for  $D$ .*

*Proof.* For any  $\epsilon > 0$ , there is some  $N$ , such that in the half-plane  $\Re s > \beta + \epsilon$  the product  $\prod_{n < N} \zeta(ms - n)^{c_{nm}}$  has the same zeros and poles as  $D(s)$ . Hence, to prove that the line  $\Re s = \beta$  is the natural boundary of  $D(s)$  it suffices to show that for every fixed  $t_0 \in \mathbb{R}$  and  $\delta > 0$  there is some  $\epsilon > 0$  such that for  $N$  sufficiently large the product  $\prod_{n < N} \zeta(ms - n)^{c_{nm}}$  has a pole or a zero in the rectangle  $R$  defined by the conditions  $\beta + \epsilon < \Re s < \beta + \delta$ ,  $t_0 < \Im s < t_0 + \delta$ . The latter would follow, if we could show that there exist integers  $n, m$  with  $c_{nm} \neq 0$  and a zero  $\rho$  of  $\zeta$ , such that  $\frac{n+\rho}{m} \in R$ , and such that for all other values  $n'$  and  $m'$ ,  $\rho' = m' \frac{n+\rho}{m} - n'$  is

not a zero of  $\zeta$ . Suppose first that  $\zeta$  has infinitely many zeros off the line  $\Re s = \frac{1}{2}$ . Then we choose one such zero  $\rho_0$  with sufficiently large imaginary part, and apply Lemma 1 with  $\mathcal{P}$  being the set of primes  $p$  such that there is some  $n$  with  $c_{np} \neq 0$  and  $n/p + 1/2p > \beta$ , and  $\mathcal{X}$  being the set of all imaginary parts of roots of  $\zeta$  of the form  $m' \frac{n+\rho_0}{m} - n'$  to obtain a contradiction as in the proof of Theorem 1.

Now suppose that up to a finite number of counterexamples, the Riemann hypothesis holds true. Since these finitely many zeros off the line of  $\zeta$  can only induce a discrete set of zeros of  $D(s)$  apart from a possible accumulation points on the real line, we can totally neglect these zeros. Similarly, we forget about all pairs  $n, m$  apart from those that can induce zeros to the right from  $\beta$ ; in particular, we may assume that  $\beta$  is the only limit point of the set of all occurring fractions  $\frac{n}{m}$ . Finally, we can neglect finitely many pairs  $n, m$  and assume that all fractions  $\frac{n}{m}$  are in an arbitrarily small interval around  $\beta$ . The contribution of a zero  $\rho$  induced by some  $c_{nm} \neq 0$  can be cancelled by a zero  $\rho'$  only if there are integers  $n', m'$  with  $m(\frac{1}{2} + i\gamma) - n = m'(\frac{1}{2} + i\gamma') - n'$ , that is,  $m\gamma = m'\gamma'$ , and  $m - 2n = m' - 2n'$ . Without loss we may assume that  $\beta \neq \frac{1}{2}$ , that is,  $\frac{n}{m} - \frac{1}{2}$  is bounded away from 0. Then the second equation implies an upper bound for  $m'$ , that is, for at each cancellation among zeros there are only finitely many zeros concerned, that is, we may assume that among these  $\rho$  is the one with largest imaginary part. But now we can apply Lemma 1 again, this time to the set of all zeros of  $\zeta$ , and obtain again a contradiction.  $\square$

### 3. A RANDOM SERIES

Although the problem to decide whether a given Dirichlet-series can be meromorphically extended to the whole complex plane may be very difficult, we believe that in most cases the obvious candidate of a natural boundary is in fact the natural boundary. This belief is strengthened by the following theorem, which shows that this conjecture is generically true. Note that our definition of a random series differs from the usual one, in which random coefficients are used (for example in Kahane [8] or Quéffelec [9]). We call a function  $f$  regular in a domain  $\Omega$ , if it is meromorphic on some branched covering of  $\Omega \setminus A$ , where  $A$  is a subset having no cluster points within  $\Omega$ , that is,  $f$  is holomorphic with the exception of poles and branch points.

added one sentence

**Theorem 3.** *Let  $(a_\nu), (b_\nu), (c_\nu)$  be real sequences, such that  $a_\nu, b_\nu \rightarrow \infty$ , and set  $\sigma_h = \limsup_{\nu \rightarrow \infty} -\frac{b_\nu}{a_\nu}$ . Let  $\epsilon_\nu$  be a sequence of independent real random variables, such that*

$$\liminf_{\nu \rightarrow \infty} \max_{x \in \mathbb{R}} P(\epsilon_\nu = x) = 0,$$

and suppose that for  $\sigma > \sigma_h$  the series

$$(1) \quad \sum_{\nu=1}^{\infty} \frac{|c_\nu + \epsilon_\nu|}{2^{a_\nu \sigma + b_\nu}}$$

converges almost surely. Then with probability 1 the function

$$Z(s) = \prod_{\nu=1}^{\infty} \zeta(a_\nu s + b_\nu)^{c_\nu + \epsilon_\nu}$$

is regular in the half-plane  $\Re s > \sigma_h$  and has the line  $\Re s = \sigma_h$  as its natural boundary.

changed holomorphic to regular

added one sentence

*Proof.* If the series (1) converges, then  $Z$  can be written as a finite product of  $\zeta$ -functions multiplied by some function which converges uniformly in the half-plane  $\Re s > \sigma_h + \epsilon$  for each  $\epsilon > 0$ . Hence,  $Z(s)$  is regular in this half-plane. Let  $s_0 = \sigma_h + it$  be a point on the supposed boundary with  $t \neq 0$  rational, and consider for a natural number  $n$  the square  $S$  with side length  $\frac{2}{n}$  centred in  $s_0$ , that is, the set  $[\sigma_h - \frac{1}{n}, \sigma_h + \frac{1}{n}] \times [t - \frac{1}{n}, t + \frac{1}{n}]$ . Let  $\epsilon > 0$  be given. We show that with probability  $> 1 - \epsilon$  the function  $Z$  is not meromorphic on  $S$ , or has a zero or a pole in  $S$ . Once we have shown this, we are done, for if  $s_0$  were an interior point of the domain of holomorphy of  $D$ , there would be some  $n$  such that  $Z$  would be holomorphic on  $S$ , and have a zero or a pole in  $S$  almost surely. Letting  $n$  tend to  $\infty$ , we see that  $s_0$  is either a pole or a zero, or a cluster point of poles or zeros. Hence, with probability 1, every point with rational imaginary part on the line  $\Re s = \sigma_h$  is a pole, a zero, or a cluster point of poles or zeros. Hence,  $\sigma_h$  is a natural boundary of  $Z$  almost surely.

To prove the existence of a pole or zero in  $S$ , note first that by the same argument used to prove almost sure convergence to the right of  $\sigma_h$ , we see that if for some  $\epsilon > 0$  there are infinitely many indices  $\nu$  with  $-\frac{b_\nu}{a_\nu} < \sigma_h - \frac{1}{n}$ , the product defining  $Z$  extended over all such indices converges uniformly in  $\Re s > \sigma_h - \frac{1}{n}$ , hence, deleting these indices does not alter our claim. In particular, we may assume that for all  $\mu$  sufficiently large we have  $|\sigma_h - \frac{a_\mu}{b_\mu}| < \frac{1}{n}$ ,  $a_\mu > 3n$ ,  $|a_\mu t| > 1000$ , as well as  $\max_{x \in \mathbb{R}} P(\epsilon_\mu = x) < \epsilon$ . For such an index  $\mu$  set

$$Z_\mu(s) = \prod_{\nu \neq \mu}^{\infty} \zeta(a_\nu s + b_\nu)^{c_\nu + \epsilon_\nu}.$$

If  $Z$  is meromorphic on  $S$ , so is  $Z_\mu$ . Let  $D_1$  be the divisor of the restriction of  $Z_\mu$  to  $S$ , and let  $D_2$  be the divisor of  $\zeta(a_\mu s + b_\mu)$  restricted to  $S$ . We have to show that  $D_1 + (c_\mu + \epsilon_\mu)D_2$  is non-trivial with probability  $> 1 - \epsilon$ . To do so, it suffices to show that  $D_2$  is non-trivial, since then  $D_1 + xD_2$  is trivial for at most one value of  $x$ , and we assumed that  $\epsilon_\mu$  is not concentrated on a single value. The preimage of  $S$  under the linear map  $s \mapsto a_\mu s + b_\mu$  is a square of side  $\ell > 6$  and centre with real part of absolute value  $\leq \ell$  and imaginary part of absolute value  $> 1000$ . Hence, the number of zeros of  $\zeta(a_\mu s + b_\mu)$  in  $S$  equals  $N(T+h) - N(T)$ , where  $N$  denotes the number of zeros of  $\zeta$  with imaginary part  $\leq T$ , and  $T$  and  $h$  are certain real numbers satisfying  $T \geq 1000$  and  $h \geq 6$ . Now Backlund [1] showed that for  $T > 1000$  we have

$$\left| N(T) - \frac{T}{2\pi} \log \frac{T}{2\pi} \right| \leq 0.7 \log T,$$

that is,  $N(T+6) > N(T)$  for  $T > 1000$ , which shows that  $D_2$  is non-trivial, and proves our theorem.  $\square$

#### 4. NATURAL BOUNDARIES AND ASYMPTOTIC FORMULAE

The hunt for natural boundaries has certainly some intrinsic interest, however, in this section we show that the existence of a natural boundary implies the non-existence of an asymptotic formula of a certain kind. This leads to a lesser known kind of  $\Omega$ -result : usually when proving an  $\Omega$ -result, one first derives an explicit formula with oscillating terms and then shows that these terms cannot cancel each other out for all choices of the parameters. Here we show that even if we allow for

infinite oscillatory sums to be part of the main terms, we still get lower bounds for the error terms.

**Theorem 4.** *Let  $a_n$  be a sequence of complex numbers, and suppose that there exist an explicit formula of the form*

$$(2) \quad A(x) := \sum_{n \leq x} a_n = \sum_{\rho \in \mathcal{R}} c_\rho x^\rho + \mathcal{O}(x^\theta),$$

where for some constant  $c$  we have  $|c_\rho| \ll (1 + |\rho|)^c$  and  $|\mathcal{R} \cap \{s : \Re s > \theta, |\Im s| < T\}| \ll T^c$ . Then the Dirichlet-series  $D(s) = \sum a_n n^{-s}$  can be meromorphically continued to the half-plane  $\Re s > \theta$ .

Since the condition  $\Re s > \theta$  describes an open set, we could have formulated this theorem with an error term  $\mathcal{O}(x^{\theta+\epsilon})$  for every  $\epsilon > 0$ , or with  $\mathcal{O}(x^{\theta-\epsilon})$  for some  $\epsilon > 0$  without affecting the conclusion. We shall move freely between these different formulations without further mention.

*Proof.* Our claim does not change if we absorb finitely many of the summands  $c_\rho x^\rho$  into the sequence  $a_n$ . Thus we can assume that all  $\rho$  satisfy  $|\Im \rho| \geq 1$ .

Set  $A_0(x) = A(x)$ ,  $A_{k+1}(x) = \sum_{\nu \leq x} A_k(\nu)$ . Then there exists an explicit formula

$$A_k(x) = x^k \sum_{\rho \in \mathcal{R}_k} c_\rho^{(k)} x^\rho + \mathcal{O}(x^\theta),$$

where  $\mathcal{R}_k$  is contained in the set of all numbers of the form  $\{\rho - j : \rho \in \mathcal{R}, j \in \mathbb{N}\}$ , and  $c_\rho^{(k+1)} = \frac{c_\rho^{(k)}}{\rho} + \mathcal{O}(\max_{j \in \mathbb{N}} c_{\rho+j}^{(k)})$ . By induction on  $k$  we obtain

$$c_\rho^{(k)} \ll \max\{|c_{\rho+j}| : j \in \mathbb{N}\} \rho^{-k + \max\{j : \rho+j \in \mathcal{R}\}},$$

where  $c_{\rho+j}$  is understood to be 0, if  $\rho + j \notin \mathcal{R}$ . Combining this estimate with the assumption on the number of elements in  $\mathcal{R}$ , we see that there exists some  $k$  such that the explicit formula for  $A_k$  converges absolutely. Note that we can immediately delete all terms with  $\Re \rho < \theta$ , and  $\Re \rho$  is bounded, since otherwise the explicit formula for  $A(x)$  would not converge in any sense. Thus, putting  $M = \lceil \sup\{\Re \rho : \rho \in \mathcal{R}\} - \theta \rceil$ , we obtain

$$c_\rho^{(k)} \ll \max\{|c_{\rho+j}| : j \in \mathbb{N}\} \rho^{-k+M}.$$

Applying partial summation and interchanging the order of summations, which is now allowed since the explicit formula is absolutely converging, we find

$$D(s) = \sum_{n \geq 1} \sum_{\rho \in \mathcal{R}_k} c_\rho^{(k)} n^{k+\rho} \Delta^{k+1} n^{-s} + R(s),$$

where  $R(s)$  is holomorphic in  $\Re s > \theta$ , and  $\Delta$  denotes the difference operator. Using Laurent expansion, we have for every  $N$  the asymptotic formula

$$\Delta^{k+1} n^{-s} = \sum_{i=0}^N a_i(s) n^{-s-k-i-1} + \mathcal{O}(n^{-s-k-N-2})$$

where the coefficients  $a_i$  are polynomials of degree  $i+k+1$ . Inserting this expression in the previous formula, we obtain

$$D(s) = \sum_{n \geq 1} \sum_{\rho \in \mathcal{R}_k^*} c_\rho^{(k)} n^{\rho-s} \left( \sum_{i=0}^N a_i(s) n^{-i-1} + \mathcal{O}(n^{-N-1}) \right).$$

Choosing  $N$  sufficiently large, the error term yields a function holomorphic in  $\Re s > \theta$ , and collecting all terms coming from one zero  $\rho$  which are independent of  $n$  into one polynomial, we obtain

$$(3) \quad D(s) = \sum_{\rho \in \mathcal{R}_k^*} P_\rho(s) \zeta(s - \rho - 1) + R^*(s),$$

where  $R^*(s)$  is holomorphic in  $\Re s > \theta - 1$ , and  $P_\rho$  is a polynomial of degree  $\leq N + k$  with coefficients  $\ll c_\rho^{(k)} \ll |\rho|^{-k+M}$ . We claim that this series is absolutely and uniformly converging in each domain of the form  $D = \{s : \Re s > \theta + \epsilon, |\Im s| < T\}$ , apart from the poles of  $\zeta$  occurring explicitly. To prove this, we first have to estimate  $|P_\rho(s)|$ . The bounds for the degree and the coefficients imply

$$|P_\rho(s)| \ll C_{M,N} (1 + |s|)^{N+k} |\rho|^{-k+M}.$$

Since we only care about convergence, we may neglect finitely many terms. Thus we restrict our considerations to zeros  $\rho$  with  $|\Im \rho| > T^2$ , that is,  $|\rho| > |s|^2$ . Finally, the functional equation for  $\zeta$  implies  $\zeta(s) \ll (1 + |\Im s|^{\max(\frac{1-\Re s}{2}, 0) + \epsilon})$ , and we obtain

$$\begin{aligned} P_\rho(s) \zeta(\rho - s) &\ll (1 + |\rho|)^{-k+M} (1 + |\Im s|^{\max(\frac{\Re s - \rho + 1}{2}, 0) + \epsilon} (1 + |s|)^{N+k}) \\ &\ll (1 + |\rho|)^{-k+M + \frac{N+k}{2} + \max(\frac{\Re s - \rho + 1}{4}, 0) + \epsilon} \\ &\ll (1 + |\rho|)^{-c-2}, \end{aligned}$$

provided that

$$k > 4 + 2c + 2M + 2N + \max\left(\frac{\Re s - \rho + 1}{2}, 0\right).$$

Hence, the terms belonging to  $\rho$  are of order  $\mathcal{O}((1 + |\rho|)^{-c-2})$ , whereas their number up to some constant  $T$  is of order  $\mathcal{O}(T^c)$ , hence, the series (3) converges absolutely and uniformly in  $D$ . Hence, it represents a function holomorphic in  $\Re s > \theta$ , with the exception of the discrete set of poles contained in  $\mathcal{R}_k^*$ . Since for sufficiently large real part the right hand side of (3) represents  $D(s)$ , we deduce that this representation yields a meromorphic continuation of  $D$  to the half-plane  $\Re s > \theta$ .  $\square$

**Corollary 2.** *Let  $a_n$  be a sequence of complex numbers such that the generating Dirichlet-series has a natural boundary at  $\Re s = \sigma_h$ . Then there does not exist an explicit formula of the form (2). In particular, for any sequence  $\alpha_i, \beta_i$ ,  $1 \leq i \leq k$  and any  $\epsilon > 0$  we have*

$$A(x) = \sum \alpha_i x^{\beta_i} + \Omega(x^{\sigma_h - \epsilon}).$$

In general, even if  $D(s)$  is meromorphic in the entire plane we cannot expect to obtain an explicit formula, since the integral taken over the shifted path of integration need not converge. For example, for the Dirichlet-divisor problem we have an  $\Omega$ -estimate of size  $x^{1/4}$ , whereas the corresponding Dirichlet-series  $\zeta^2(s)$  is meromorphic on  $\mathbb{C}$ . However, we can obtain explicit formulae after attaching a sufficiently smooth weight function. To do so, we need some bounds on the growth of the Dirichlet-series in question.

**Lemma 2.** *Let  $W \in \mathbb{Z}[X, Y]$  be a polynomial with  $W(0, 0) = 1$  and not containing the monomial  $X$ . Let  $D(s) = \prod_p W(p^{-1}, p^{-s})$  be the associated Dirichlet-series, and let  $\sigma_o$  be the abscissa of obvious meromorphic continuation, and let  $\sigma > \sigma_o$  be a real number.*



- (1) There exists a  $\mu(\sigma)$  such that  $D(s)$  is the quotient of two functions  $f_1, f_2$ , both of which are holomorphic in the half-plane  $\Re s \geq \sigma_o$  up to a bounded number of poles on the real axis, and satisfy  $|f_i(\sigma + it)| \ll |t|^{\mu(\sigma)}$  for  $|t| > 1$ .
- (2) The number of poles of  $D$  in the domain  $\Re s \geq \sigma$ ,  $|\Im s| \leq T$  is bounded above by  $c_\sigma T \log T$ .
- (3) There is some  $\mu^*(\sigma)$ , such that for every  $\epsilon > 0$  and  $T$  sufficiently large there exists a path  $\gamma : [0, 1] \rightarrow \mathbb{C}$  consisting of horizontal and vertical lines only, which is contained in the strip  $\sigma \leq \Re s \leq \sigma + \epsilon$ , has length  $\leq (2 + \epsilon T)$ , such that  $\Im \gamma(0) = -T$ ,  $\Im \gamma(1) = T$ , and  $|D(s)| < e^{\mu^*(\sigma) \log^2 T}$  on all of  $\gamma$ .

Note that the third statement is an adaptation of a result due to Turán [7, Appendix G].

*Proof.* For each  $\sigma > \sigma_o$ , there exists a finite product of the form  $D^*(s) = \prod_{\kappa=1}^k \zeta(a_\kappa s + b_\kappa)^{c_\kappa}$ , such that  $D(s) = D^*(s)R(s)$  with  $R(s)$  holomorphic and bounded in the half-plane  $\Re s > \sigma$ . Collecting terms with  $c_\kappa > 0$  in  $f_1$ , and terms with  $c_\kappa < 0$  in  $f_2$ , the first statement follows from the fact that  $\zeta(s)$  grows only polynomially in each strip of bounded width. Moreover, the number of poles of  $D^*$  in the region  $\Re s \geq \sigma$ ,  $|\Im s| \leq T$  is bounded above by some multiple of the number of zeros of  $\zeta(s)$  in the domain  $\Re s > 0$ ,  $|\Im s| \leq T \max_k a_k$ , which implies the second assertion. For the third note that for each  $s$  with  $\Re s > \sigma$  we have

$$\frac{D^{*'}}{D^*}(s) = \sum_{\rho} \frac{m_{\rho}}{s - \rho} + \mathcal{O}(\log T),$$

where the sum runs over all poles and zeros of  $D^*(s)$  with  $|\Im(\rho - s)| < \epsilon$ , and  $m_{\rho}$  is the (signed) multiplicity of the pole  $\rho$ . The same argument when used to prove the second assertion also yields that for  $|T| > 2$  the number of poles and zeros  $\rho$  of  $D^*$  with  $T \leq \Im \rho \leq T + 1$  is  $\leq c_\sigma |T|$ , hence, there is some  $\sigma' \in [\sigma, \sigma + \epsilon]$ , such that there is no pole or zero  $\rho$  of  $D$  with  $T \leq \Im \rho \leq T + 1$  and  $|\Re \rho - \sigma'| < \frac{\epsilon}{c_\sigma \log T}$ . Hence, on this line segment, we have  $|\frac{D^{*'}}{D^*}| \ll \log^2 T$ . Choosing  $T$  in such a way that  $D^*$  has no poles or zeros in the half-strip  $\Re s > \sigma$ ,  $|T - \Im s| < \frac{\epsilon}{c_\sigma \log T}$ , we find that there exists a path  $\gamma$  as desired such that each point on  $\gamma$  can be linked to a point in the half plane of absolute convergence of  $D$  by a path of length  $\ll 1$ , such that  $|\frac{D^{*'}}{D^*}| \ll \log^2 T$  on this path. Hence, we deduce  $D(s) < e^{\mu^*(\sigma) \log^2 T}$  on  $\gamma$ .  $\square$

Now we give an example. In [4] we found a bijection between right cosets of  $2t \times 2t$  symplectic matrices and submodules of finite index of  $\mathbb{Z}^{2t}$  which are equal to their duals and which we call polarised. The counting function obtained corresponds to the  $p$ -adic zeta function of Weil-Igusa and occurs, for example, in [5].

**Theorem 5.** Denote by  $a_n$  the number of polarised submodules of  $\mathbb{Z}^6$  of order  $n$ . Then we have for every  $\epsilon > 0$

$$(4) \quad A(x) := \sum_{n \geq 1} a_n e^{-n/x} = c_1 x^{7/3} + c_2 x^2 + c_3 x^{5/3} + \sum_{\rho} \alpha_{\rho} x^{\frac{\rho+8}{6}} + \mathcal{O}(x^{4/3+\epsilon}),$$

where  $\rho$  runs over all zeros of  $\zeta$ , and the coefficients  $c_1, c_2, c_3$ , and  $\alpha_{\rho}$  are numerically computable constants. More precisely, we have  $c_1 = 2.830\dots$ ,  $c_2 = 1.168\dots$ , and  $c_3 = 0.1037\dots$ . Moreover, the error term cannot be improved to  $\mathcal{O}(x^{4/3-\epsilon})$  for any fixed  $\epsilon > 0$ .

*Proof.* The generating function for  $a_n$  has the form [4]

$$\begin{aligned} Z(s/3) &= \zeta(s)\zeta(s-3)\zeta(s-5)\zeta(s-6) \prod_p \left(1 + p^{1-s} + p^{2-s} + p^{3-s} + p^{4-s} + p^{5-2s}\right) \\ &= \zeta(s)\zeta(s-3)\zeta(s-5)\zeta(s-6) \frac{\zeta(s-4)}{\zeta(2s-8)} \\ &\quad \times \prod_p \left(1 + \frac{p^{1-s} + p^{2-s} + p^{3-s} + p^{5-2s}}{1 + p^{4-s}}\right) \end{aligned}$$

and in [5] it was proved that  $\Re s = \frac{4}{3}$  is the natural boundary for the above. The product over primes converges absolutely and uniformly in every half-plane  $\Re s > \frac{4}{3} + \epsilon$ . Hence,  $Z(s)$  has simple poles at  $7/3$ ,  $2$  and  $5/3$ , poles at the zeros of  $\zeta(6s-8)$ , and no other singularities in the half plane  $\Re s > 4/3$ . Applying the Mellin transform

$$e^{-y} = \int_{3-i\infty}^{3+i\infty} \Gamma(s)y^s ds$$

we obtain

$$A(x) = \frac{1}{2\pi i} \int_{3-i\infty}^{3+i\infty} Z(s)\Gamma(s)x^s ds.$$

For  $\sigma$  and  $\epsilon > 0$  fixed, we have  $\Gamma(\sigma + it) \ll e^{-(\frac{\sigma}{2}-\epsilon)t}$ . We now choose a path as in Lemma 2, and shift the integration to this path. Due to the rapid decrease of  $\Gamma$ , we find that for  $T = \log^3 x$  the integral on the new path is bounded above by  $x^{4/3+\epsilon}$ . Hence, we obtain the formula

$$A(x) = \sum_{\Re \rho > 4/3+\epsilon} \Gamma(\rho)x^\rho \text{res}_{s=\rho} Z(s) + \mathcal{O}(x^{4/3+\epsilon}),$$

where  $\rho$  runs over  $7/3$ ,  $2$ ,  $5/3$ , and all complex numbers  $4/3 + \rho/6$ , where  $\rho$  runs over all non-trivial zeros of  $\zeta$ .

To compute the values of  $c_1, c_2, c_3$ , we only have to compute the residuum of  $Z(s)$  at these points, which does not pose any problems, since the Euler products involved converge rather fast. We obtain the residue  $2.377$ ,  $-1.168$ ,  $0.1149$ , respectively, which yields the constants mentioned in the Theorem. Using Mathematica, CPU-time for these computations was about 30 seconds.

In view of [5],  $Z(s)$  has a natural boundary on the line  $\Re s = 4/3$ , hence, the proof of the  $\Omega$ -result runs parallel to the proof of Theorem 4.  $\square$

Having an explicit formula, we can use standard methods to draw conclusions from it. For example, we have the following.

**Corollary 3.** *Define  $A(x)$  as above. Then we have*

$$A(x) = c_1 x^{7/3} + c_2 x^2 + c_3 x^{5/3} + \mathcal{O}\left(x^{3/2} e^{-c \frac{\log x}{(\log \log x)^{2/3+\epsilon}}}\right)$$

and

$$A(x) = c_1 x^{7/3} + c_2 x^2 + c_3 x^{5/3} + \Omega_\pm(x^{17/12-\epsilon})$$

*Proof.* Note that apart from the poles at  $7/3, 2, 5/3$  and  $3/2$  all singularities of  $D(s)$  in the half-plane  $\Re s > 4/3$  come from zeros of  $\zeta(6s - 8)$ , hence, for a certain constant  $c$  we have for all  $\rho$  occurring in (4) the relation  $\Re \rho < \frac{3}{2} - \frac{c}{(\log \log |\Im s|)^{2/3+\epsilon}}$ . Since  $\Gamma(s)$  decreases exponentially fast on each line parallel to the imaginary axis, we see that the contribution of a single zero is at most

$$\max_{T>3} x^{\frac{3}{2} - \frac{c}{\log^{2/3+\epsilon} T}} e^{-c'T} \ll x^{\frac{3}{2}} e^{-c \frac{\log x}{(\log \log x)^{2/3+\epsilon}}};$$

moreover, the contribution of zeros with imaginary part  $> \log^2 T$  is negligible. Hence, the contribution of all zeros of  $\zeta(6s - 8)$  and the error term in (4) together give an error term of order  $x^{\frac{3}{2}} e^{-c \frac{\log x}{(\log \log x)^{2/3+\epsilon}}}$ , and our claim follows.

The  $\Omega$ -estimate follows from a standard application of Turán's theory of power-sums, confer [7, Chapter 47].  $\square$

Of course, these computations did not make use of the natural boundary of  $Z$ , however, the existence of a natural boundary implies that there is a limit to what can be achieved by complex analytic means.

## REFERENCES

- [1] R. Backlund, Über die Nullstellen der Riemannschen Zetafunktion, *Acta Math.* **41** (1918), 345–375.
- [2] T. Estermann, On certain functions represented by Dirichlet series, *Proc. London Math. Soc.* **27** (1928), 435–448.
- [3] G. Dahlquist, On the analytic continuation of Eulerian products, *Ark. Mat.* **1** (1952), 533–554.
- [4] G. Bhowmik, F. Grunewald, Counting Functions of Classical Groups, preprint.
- [5] M. du Sautoy, F. Grunewald, Zeta functions of groups: zeros and friendly ghosts, *Amer. J. Math.* **124** (2002), 1–48.
- [6] M. du Sautoy, Zeta functions of groups and natural boundaries, preprint at <http://www.maths.ox.ac.uk/~dusautoy/>
- [7] P. Turán, *On a new method of analysis and its applications*, Wiley-Interscience, New York, 1984.
- [8] J.P. Kahane, *Some random series of functions*, CUP, 1985.
- [9] H. Quéffelec, Propriétés presque sûres des séries de Dirichlet et des produits d'Euler, *Can.J.Math.* **32**, 1980, 531–558.

Gautami Bhowmik, Université de Lille 1, Laboratoire Paul Painlevé, U.M.R. CNRS 8524, 59655 Villeneuve d'Ascq Cedex, France <b>bhowmik@math.univ-lille1.fr</b>	Jan-Christoph Schlage-Puchta, Albert-Ludwigs-Universität, Mathematisches Institut, Eckerstr. 1, 79104 Freiburg, Germany <b>jcp@math.uni-freiburg.de</b>
---	---

Note added in proof

In the statement of Theorem 3, the term holomorphic has to be replaced by meromorphic upto branch points, that is, meromorphic on a branched covering with branch points having no cluster point in the interior of the half-plane  $\Re s > \sigma_h$ . In fact, the finite product of  $\zeta$ -functions dominating the behaviour of  $Z$  in a half-plane  $\Re s > \sigma_h + \epsilon$  can yield branch points at all poles and zeros of the involved  $\zeta$ -functions.

We would like to thank J. P. Kahane for pointing this out to us.