THE MAXIMAL DOMAIN OF MEROMORPHIC CONTINUATION OF A DIRICHLET-SERIES

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1. Introduction

Let $D(s)=\sum_{n\geq 1}a_n n^{-s}$ be a Dirichlet-series, such that $a_n\neq 0$ infinitely often, and $\limsup_{n:a_n\neq 0}\frac{\log|a_n|}{\log n}+1=\sigma_0\neq\pm\infty$. Then D converges uniformly in the half-plane $\Re s>\sigma_0$.

In examples usually encountered the Dirichlet series is either meromorphically continuable to the whole complex plane or its maximal domain of meromorphic continuation is a half plane.

A classical result is due to Estermann[3] who showed that the Dirichlet-series $D(s)=\prod_p W(p^{-s})$ with $W(x)$ an integer valued polynomial when not meromorphically continuable to the whole complex plane has the half-plane to the left of $\Re s=0$ as its maximal domain of meromorphic continuation. Many extensions of this result followed. More recently the maximal domain of meromorphic continuationof Dirichlet series arising from different contexts like counting rational points on algebraic varietes have been popular. Thus De la Breteche and Swinnerton-Dyer [1] proved that the height zeta function associated to singular cubic surface $x_1x_2x_3=x_4^3$, has a natural boundary at $\Re s=3/4$, by considering the Euler-product corresponding to the rational function

$$W(X,Y)=1+(1-X^3Y)(X^6Y^{-2}+X^5Y^{-1}+X^4+X^2Y^2+XY^3+Y^4)-X^9Y^3.$$

Another spark of interest came from the study of analytic properties of group zeta functions The local zeta function associated to the algebraic group $\mathcal{G}$ is defined as

$$Z_p(\mathcal{G},s)=\int_{\mathcal{G}_p^+}|\det(g)|_p^{-s}\,d\mu$$

where $\mathcal{G}_p^+=G(\mathbb{Q}_p)\cap M_n(\mathbb{Z}_p)$, $|\ .\ |_p$ denotes the p-adic valuation and μ is the normalised Haar measure on $\mathcal{G}(\mathbb{Z}_p)$. In particular the zeta function associated to the group $\mathcal{G}=GSp_6$ given by

$$Z(s/3)=\zeta(s)\zeta(s-3)\zeta(s-5)\zeta(s-6)\prod_p\Big(1+p^{1-s}+p^{2-s}+p^{3-s}+p^{4-s}+p^{5-2s}\Big)$$

has the natural boundary of meromorphic continuation $\Re\ s=4$ (du Sautoy and Grunewald[6]).

It is in the context of zeta functions of groups that Du Sautoy and Woodward [7], Pg 153, asked whether the maximal domain of meromorphic continuation of D is always a half-plane. Here we give a negative answer to this question. In fact, we show that every subset of the complex plane, which satisfies the obvious restrictions occurs as maximal domain of holomorphic continuation of a Dirichlet-series.

Theorem 1. *Let $\Omega \subseteq \mathbb{C}$ be an open connected set, and σ_0 be a real number, such that $\{s : \Re s \leq \sigma_0\} \subseteq \Omega$. Then there exists a Dirichlet-series $D(s) = \sum a_n n^{-s}$, which is holomorphic in Ω, has simple poles in every isolated point of $\mathbb{C} \setminus \Omega$, is absolutely convergent precisely in the half-plane $\Re s > \sigma_0$, and cannot be meromorphically continued into any larger domain.*

This result may not be completely satisfactory, since the Dirichlet series constructed lacks any structure such as an Euler product. Restricting Ω we obtain examples which do have an Euler product.

Theorem 2. *Let $\Omega \subseteq \mathbb{C}$ be an open connected set, and assume that*

$$\{s : \Re s > 1\} \subseteq \Omega \subseteq \{s : \Re s > \frac{5}{18}\}.$$

Then there exists a Dirichlet-series $D(s) = \sum a_n n^{-s}$, which has an Euler product, is holomorphic in Ω, has a pole in every isolated point of $\mathbb{C} \setminus \Omega$, and cannot be meromorphically continued into any larger domain.

Again this result leaves something to be desired, as the different factors of the product representation cannot be described in a uniform way. However, if we assume the Riemann hypothesis and further restrict the shape of Ω we obtain a uniform result.

Theorem 3. *Assume the Riemann hypothesis. Let Ω be a simply connected open set, such that*

$$\{s : \Re s \leq \frac{1}{2}\} \subseteq \Omega \subseteq \{s : \Re s < 1\}.$$

Then there exists a function h, holomorphic in $|z| > 1$, bounded on the positive real axis, such that the Dirichlet series given by the Euler product $D(s) = \prod_p \big(1 + h(p)p^{-s}\big)$ is holomorphic in Ω, and cannot be meromorphically continued into any larger domain.

2. Preparations

For our proof we have to define a continuous path γ from some point $p \in \Omega$ to a point $q \in \partial\Omega$. Unfortunatelly in general this is impossible. For example, if $\Omega = \{s : \Re s > 0\} \setminus \{x + iy : 0 \leq x \leq 1, y = \frac{1}{x} \sin \frac{1}{x}\}$, then for s on the imaginary axis there exists no path $\gamma : [0, 1] \to \mathbb{C}$ with $\gamma(0) = 2$, $\gamma(1) = s$, and $\gamma(t) \in \Omega$ for $0 < t < 1$. In this section we will remedy this problem.

Let X be a topological space, $O \subset X$ open, p a point in O. We call a point $q \in \partial O$ reachable, if there exists a continuous path $\gamma : [0, 1] \to X$ with $\gamma(0) = p$, $\gamma(1) = q$, and $\gamma(t) \in O$ for $0 < t < 1$. Then we have the following.

Lemma 1. *Assume that X is pathwise and locally pathwise connected, and that O is pathwise connected. Then the set of reachable points is dense in ∂C.*

Proof. We have to show that every open set U which contains a point q of ∂O contains a reachable point q'. By assumption U contains a pathwise connected neighbourhood of q, reducing U we may assume that U itself is pathwise connected. Since $q \in \partial O$, there exists a point $r \in U \cap O$. Since O as well as U are pathwise connected, there exist paths γ_1, γ_2 with $\gamma_1(0) = p$, $\gamma_1(1) = \gamma_2(0) = r$, $\gamma_2(1) = q$, such that γ_1 is contained in O, and γ_2 is contained in U. Since $\gamma_2^{-1}(X \setminus O)$ is closed, it contains a least element t_0. Clearly $\gamma_2(t_0) \in \partial C$. We can combine γ_1 with $\gamma_2|_{[0,t_0]}$

to obtain a path γ from p to $\gamma_2(t_0)$, such that $\gamma(t) \in X \setminus C$ for $0 < t < 1$. Since $\gamma(1) = \gamma_2(t_0) \in \partial C$ we obtain that $\gamma(t_0)$ is reachable, and by the construction it is contained in U. Hence our claim follows. $\square$

We further need some information about prime numbers, which are contained in the following. Denote by p_n the n-th prime number. Heath-Brown[4] proved the following.

Theorem 4. *We have* $\sum_{p_n \leq x} (p_{n+1} - p_n)^2 \ll x^{23/18+\epsilon}$.

From this we deduce the following.

Lemma 2. *There exists a Dirichlet-series $Z(s)$ such that the coefficients of Z are supported on primes only, $Z(s) - \zeta(s)$ is holomorphic in the half plane $\Re\, s > \frac{5}{18}$, and $|Z(s) - \zeta(s)| \ll (1 + \Im\, s)^2$ holds true uniformly in any half plane $\Re\, s > \frac{5}{18} + \epsilon$.*

Proof. Define the function $\widetilde{\Lambda}(n)$ as

$$\widetilde{\Lambda}(n) = \begin{cases} p_k - p_{k-1}, & n = p_k \\ 0, & n \text{ not prime} \end{cases},$$

and define $Z(s) = \sum_p \frac{\widetilde{\Lambda}(n)}{n^s}$. Put $S_1(x) = \sum_{n \leq x} (\widetilde{\Lambda}(n) - 1)$. Then $S_1(p) = 0$ holds true for all prime numbers p, and $S_1(n) \leq p_k - p_{k-1}$ for $p_{k-1} \leq n \leq p_k$. Now put $S_2(x) = \sum_{n \leq x} S_1(n)$. Then we have

$$S_2(x) \leq \sum_{p_n \leq x} (p_{n+1} - p_n)^2 \ll x^{23/18} \log^{1000} x,$$

By partial summation we obtain for $\sigma > 1$

$$Z(s) - \zeta(s) = \sum_{n=1}^{\infty} \frac{\widetilde{\Lambda}(n) - 1}{n^s} = \sum_{n=1}^{\infty} S_2(n) \left(\frac{1}{(n+2)^s} - \frac{2}{(n+1)^s} + \frac{1}{n^s} \right).$$

The sum on the right converges absolutely in every half-plane of the form $\sigma > \frac{5}{18} + \epsilon$, thus the abscissa of convergence of the Dirichlet series $\sum \frac{\widetilde{\Lambda}(n) - 1}{n^s}$ is $\geq \frac{5}{18}$. Since the upper bound for $|Z(s) - \zeta(s)|$ is true for all Dirichlet series in their half plane of convergence, see e.g. [5, Theorem 1.5], the proof of the statement is complete. $\square$

3. Proof of the theorem

Note first that if $\mathbb{C} \setminus \Omega$ is finite, then some linear combination of shifted Riemann ζ-functions has the required properties, hence, from now on we assume that $\mathbb{C} \setminus \Omega$ is infinite. Then we can choose a sequence (z_n) of points satisfying the following conditions.

(1) $\{z_n\}$ is dense in $\partial\Omega$;
(2) z_n is reachable for all n;
(3) For n sufficiently large we have $\Re\, z_n > -\sqrt{n}$, $|\Im\, z_n| < n$.
(4) For $n \neq m$ we have $z_n \neq z_m$.

By assumption $\mathbb{C} \setminus \Omega$ is connected, hence this set is pathwise connected, and each point of $\partial\Omega$ is in the closure of $\mathbb{C} \setminus \Omega$. Therefore for each n we can choose a path $\gamma_n : [0, 1] \to \mathbb{C}$, such that γ_n is continuous, $\gamma_n(0) = \sigma_0 + 1$, $\gamma_n(1) = z_n$, and

$\gamma_n([0,1)) \cap \Omega = \emptyset$. For $m \neq n$ we have that $t \mapsto |z_m, \gamma_n(t)|$ is continuous, positive, and defined on a compactum, hence,

$$d_n := \min(1, \min_{m<n} \min_{t\in[0,1]} \frac{|z_n - \gamma_m(t)|}{|z_m - \gamma_m(t))})$$

is a positive real number, and so is $\delta_n = 3^{-n} \prod_{\nu=1}^n d_n$. Note that $\delta_n \leq 3^{-n}$.

Now define the function

$$D(s) = \sum_{m=1}^\infty \delta_m \zeta(s - z_m + 1) + L(s - \sigma_0 + 1, \chi), \tag{1}$$

where ζ is the Riemann ζ-function, and $L(s, \chi)$ is a Dirichlet L-function to some non-principal character χ. We now prove several properties of D.

We can represent D as a Dirichlet-series with abscissa of absolute convergence equal to σ_0. We have formally

$$D(s) = \sum_{m=1}^\infty \delta_m \sum_{n=1}^\infty + \sum_{n=1}^\infty \chi(n) n^{1-\sigma_0} n^{-s} = \sum_{n=1}^\infty \Big(\chi(n) n^{1-\sigma_0} + \sum_{m=1}^\infty \delta_m n^{z_m - 1}\Big) n^{-s}.$$

We have $n^{z_m} \leq n^{\sigma_0}$, hence the sum $\sum_{m=1}^\infty \delta_m n^{z_m-1}$ converges absolutely. Moreover, we have

$$|a_m| \leq n^{\sigma_0 - 1} + \sum_{m=1}^\infty \delta_m n^{\Re z_m - 1} \leq n^{\sigma_0-1}\big(1 + \sum_{m=1}^\infty 2^{-m}\big) = 2n^{\sigma_0-1},$$

hence, the new series converges absolutely for $\Re s > \sigma_0$, and uniformly in every half-plane of the form $\Re s > \sigma_0 + \epsilon$. On the other hand we have that $\zeta(s)$ is uniformly bounded in every half-plane of the form $\Re s > 1 + \epsilon$, hence, the series $\sum_{m=1}^\infty \delta_m \zeta(s - z_m + 1)$ also converges absolutely and uniformly in every half-plane of the form $\Re s > \sigma_0 + \epsilon$, hence, the two series represent the same function, and we obtain that D is a Dirichlet-series converging in the half-plane $\Re s > \sigma_0$.

To see that the abscissa of convergence cannot be smaller than σ_0 note that

$$|a_n| \geq n^{\sigma_0-1} - \sum_{m=1}^\infty |\delta_m n^{\Re z_m - 1}| \geq n^{\sigma_0 - 1}\big(1 - \sum_{m=1}^\infty 3^{-m}\big) = \frac{1}{2} n^{\sigma_0 - 1},$$

thus the series representing D does not converge absolutely at σ_0. Note that this is the only reason to include L into the definition of D.

D can be holomorphically continued to $\mathbb{C} \setminus \Omega$. Let s_0 be a point in $C \setminus \Omega$. Since Ω is closed, there exists some $\epsilon \in (0,1)$, such that $B_{2\epsilon}(s_0) \subseteq \mathbb{C} \setminus \Omega$. We show that the series (1) represents a function holomorphic in $B_\epsilon(s_0)$. Since each summand is holomorphic in $\mathbb{C} \setminus \Omega$ it suffices to show that the series converges uniformly on $B_\epsilon(s_0)$. Consider first all indices n satisfying $|z_n - s_0| < 2$. We have $|\zeta(s)| < 2 + \frac{1}{\epsilon}$ for $\epsilon < |s-1| \leq 1$, hence, the sum over these indices is uniformly bounded. Now consider the sum over the remaining points. We have $|\zeta(\sigma + it)| < C(2 + |t|)^{\max(0, (1-\sigma)/2)+\epsilon}$, hence, if m is sufficiently large and satisfies $|z_m - s_0| \geq 2$ we have

$$\begin{aligned} \delta_m |\zeta(s - z_m + 1)| &\leq C 2^{-m} (2 + |\Im(z_m - s_0)|)^{\max(0, \frac{1 - \Re(s_0 - z_m)}{2}) + \epsilon} \\ &\leq 2^{-m}(2+m)^{\sqrt{m}}, \end{aligned}$$

and the sum over $2^{-m}(2+m)^{\sqrt{m}}$ is clearly convergent. Hence $D(s)$ can be holomorphically continued to $\mathbb{C} \setminus \Omega$.

D cannot be meromorphically continued beyond $\mathbb{C}\setminus\Omega$. We show for every n that as z approaches z_n along γ_n, then $|D(z)|$ tends to infinity. This clearly implies our claim. Choose $\epsilon > 0$ in such a way that $B_{2\epsilon}(z_n)$ contains no z_m with $m < n$ and let $\delta > 0$ be so small, that $\gamma_n([1-\delta, 1]) \subseteq B_\epsilon(z_n)$. Then the sum over $m < n$ in (1) is uniformly bounded in $B_\epsilon(z_n)$, in particular it is uniformly bounded on $\gamma_n([1-\delta, 1])$. Repeating the argument we used to show that D is holomorphic in $\mathbb{C}\setminus\Omega$ we see that the sum over all m with $|z_m - z_n| > 2\epsilon$ converges to a function holomorphic in $B_\epsilon(z_n)$. Hence we obtain for $s \in B_\epsilon(s_0)\setminus\Omega$

$$\begin{aligned} D(s) &= \delta_n\zeta(s-z_n+1) + \sum_{\substack{m>n\\ |z_m-z_n|<2\epsilon}} \delta_m\zeta(s-z_m+1) + G(s) \\ &= \frac{\delta_n}{s-z_n} + \sum_{\substack{m>n\\ |z_m-z_n|<2\epsilon}} \frac{\delta_m}{s-z_m} + H(s), \end{aligned}$$

where G and H are holomorphic in $B_\epsilon(z_n)$. For $s \in \gamma_n([1-\delta, 1])$ we therefore obtain

$$\begin{aligned} |D(s)| &\geq \frac{\delta_n}{|s-z_n|} - \sum_{\substack{m>n\\ |z_m-z_n|<2\epsilon}} \frac{\delta_m}{|s-z_m|} \\ &\geq \frac{\delta_n}{|s-z_n|} - \sum_{\substack{m>n\\ |z_m-z_n|<2\epsilon}} \frac{\delta_m}{|s-z_n|} \max_{t\in[0,1]} \frac{|\gamma(t)-z_n|}{|\gamma(t)-z_m|} \\ &\geq \frac{\delta_n}{|s-z_n|} - \sum_{\substack{m>n\\ |z_m-z_n|<2\epsilon}} \frac{\delta_m}{d_m|s-z_n|} \\ &= \frac{\delta_n}{|s-z_n|}\left(1 - \sum_{\substack{m>n\\ |z_m-z_n|<2\epsilon}} 3^{n-m}\prod_{\nu=n}^{m-1} d_\nu\right) \\ &\geq \frac{\delta_n}{|s-z_n|}(1 - \sum_{m>n} 3^{n-m}) \\ &= \frac{\delta_n}{2|s-z_n|}. \end{aligned}$$

Hence, as $s \to z_n$ along γ_n we have $|D(s)| \gg \frac{1}{|s-z_n|}$, thus z_n is a singularity of D. Since the set of singularities is dense in $\partial\Omega$, we find that D cannot be extended meromorphically beyond Ω.

4. Dirichlet series with Euler products

Note that the only properties of ζ we used are the fact that ζ is holomorphic in $\mathbb{C}\setminus\{1\}$, and that ζ has at most polynomial growth in vertical strips. We can use the functions $Z(s)$ defined in Lemma 2 or the function $\frac{\zeta'}{\zeta}(s)$ in place of ζ. These functions can only be extended into the half planes $\Re\, s > \frac{1}{2}$ and $\Re\, s > \frac{5}{18}$, respectively, however, these restrictions are irrelevant if Ω is contained in these half planes.

Hence by repeating the proof in the previous section with Z in place of ζ and using the fact that $\Omega \subseteq \{s : \Re\, s > \frac{5}{18}\}$ we obtain a Dirichlet series D, which satisfies the conditions of Theorem 1, and is supported on the set of primes. Write $D(s) = \sum \frac{a_p}{p^s}$. Then for $\Re\, s > 1$ we have that

$$\exp(D(s)) = \prod_p \exp\left(\frac{a_p}{p^s}\right) = \prod_p \sum_{k\geq 0} \frac{a_p^k}{k! p^{sk}}$$

can be represented by a Dirichlet series with an Euler product, and the left hand side is continuable to Ω and not beyond. Now $\exp(D(s))$ has an essential singularity in each isolated point of $\mathbb{C}\backslash\Omega$, since the exponential function applied to a pole yields an essential singularity.

For Theorem 3 we repeat the construction used for Theorem 2 with $\frac{\zeta'}{\zeta}(s)$. Note that we need the Riemann Hypothesis to ensure that $\frac{\zeta'}{\zeta}$ is holomorphic in $\Re\, s > \frac{1}{2}$.

Let $D(s)$ be the resulting Dirichlet series. Then $D(s)$ is supported on the set of prime powers. We now take $\exp\left(\int D(s)\right)$, which is well defined, since Ω is simply connected. Define $h(z) = \sum_{\nu=1}^\infty \delta_\nu z^{s_\nu - 1}$. Then each summand is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$, and the series converges uniformly in $|z| > 1$, since $\{s : \Re\, s > 1\} \subseteq \Omega$, and therefore $\Re\, s_\nu - 1 \leq 0$ for all n. We conclude that $h(z)$ is holomorphic in $|z| > 1 \setminus (-\infty, 0]$, bounded on the real axis, and we have

$$\begin{aligned}
\exp\left(\int D(s)\right) &= \prod_p \exp\Big(\sum_{\kappa\geq 1} \frac{p^{-\kappa s}}{\kappa} \sum_{\nu=1}^\infty \delta_\nu p^{\kappa(s_\nu - 1)}\Big)\\
&= \prod_p \left(1 + h(p)p^{-s} + \mathcal{O}(p^{-2s})\right)\\
&= \prod_p \left(1 + h(p)p^{-s}\right)G(s),
\end{aligned}$$

where $G(s)$ is holomorphic in $\Re\, s > \frac{1}{2}$ and uniformly bounded in each half plane $\Re\, s > \frac{1}{2} + \epsilon$. Hence $\prod_p \left(1 + h(p)p^{-s}\right)$ has the same domain of holomorphic continuation as $\exp\left(\int D(s)\right)$, and our claim follows.

5. Explicit examples

If the boundary of Ω is not too wild, the argument can be simplified. For example, if there is some $\epsilon > 0$ such that for every $z_0 \in \partial\Omega$ there is an $\delta > 0$ and some $\alpha \in [0, 2\pi]$, such that the set $\{|z - z_0 < \delta, |\arg(z - z_0) - \alpha| < \epsilon\}$ meets Ω only in z_0, then we can choose the paths γ_n in such a way that the final bit of this path follows the ray $\arg(z - z_0) = \alpha$, and we do not have to worry about the δ_n anymore. In particular, let $\varphi : \mathbb{R} \to \mathbb{C}$ is differentiable with non-vanishing derivative, which describes a Jordan curve on the complex sphere. Assume that one of the connected components of $\mathbb{C} \setminus \varphi(\mathbb{R})$ contains a right half-plane, and call this domain Ω. Then we can explicitly write down a Dirichlet series D which has Ω as maximal domain of meromorphic continuation by choosing $\varphi(\mathbb{Q})$ as a countable dense set of $\partial\Omega$.

As an example consider the domain $\Omega = \{z : (\Re\, z)^2 > -(\Im z)^3\}$, that is, the points to the right of the (singular) curve $y^2 = -x^3$. We can parametrize this curve as $\varphi(t) = (-|t|^{2/3}, t)$, and $\varphi(\mathbb{Q})$ is obviously dense in $\partial\Omega$. Now consider the series

$$D_\Omega(s) = \sum_{p=-\infty}^{\infty} \sum_{q=1}^{\infty} 2^{-|p|+q} \zeta\Big(s + 1 + \big|\frac{p}{q}\big|^{2/3} - \frac{p}{q} i\Big).$$

It is easy to see that this series has Ω as maximal domain of meromorphic as well as holomorphic continuation. Note that the map $\mathbb{Z} \times \mathbb{N} \to \mathbb{Q}$ given by $(p, q) \mapsto \frac{p}{q}$ is not injective, however, this does not seriously affect the behaviour of the series. Develloping D_Ω as a Dirichletseries we obtain

$$D_\Omega(s) = \sum_{n=1}^{\infty} n^{-s} \sum_{p=-\infty}^{\infty} \sum_{q=1}^{\infty} 2^{-|p|+q}\, n^{-|p/q|^{2/3} - pi/q}$$

we see that the coefficients are represented by well converging series.

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