AN ADDITIVE PROPERTY OF ALMOST PERIODIC SETS

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ABSTRACT. We show that a set is almost periodic if and only if the associated exponential sum is concentrated in the minor arcs. Hence binary additive problems involving almost periodic sets can be solved using the circle method. This equivalence is used to give simple proofs of theorems of J. Brüdern.

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A function $f : \mathbb{N} \to \mathbb{C}$ is called $\mathcal{B}^2$-almost periodic, if there is a sequence of periodic functions f_q, such that

$$\lim_{q \to \infty} \limsup_{x \to \infty} \frac{1}{x} \sum_{n \leq x} |f(n) - f_q(n)|^2 = 0$$

A set is called $\mathcal{B}^2$-almost periodic, if its characteristic function is $\mathcal{B}^2$-almost periodic. The theory of almost periodicity is quite rich, see e.g. [3].

Let $\mathcal{N}$ be a set of integers. $\mathcal{N}$ is called distributed, if for any q and a, the density $f(q,a) = \lim_{x \to \infty} \frac{1}{x} \#\{n \leq x, n \in \mathcal{N}, n \equiv a \pmod{q}\}$ exists. A set $\mathcal{N}$ is called extremal, if it has positive density ρ, is distributed, and we have

$$\frac{1}{\rho} = \sum_{q=1}^{\infty} \sum_{(a,q)=1} \left| \sum_{b=1}^{q} \frac{f(q,a)}{\rho} e\left(\frac{ab}{q}\right) \right|^2$$

Note that the sum over a runs over residue classes prime to q, whereas the sum over b runs over all residue classes. Especially, if $q > 1$ and $f(a,q)$ does not depend on a, the inner sum vanishes. This definition is motivated by additive number theory. It turns out that binary additive problems involving extremal sets can be solved using the circle method. Define the major arcs $M(x,Q) = \bigcup_{q \leq Q} \bigcup_{(a,q)=1} \left[\frac{a}{q} - \frac{Q}{x}, \frac{a}{q} + \frac{Q}{x} \right]$ and the minor arcs $m(x,Q) = [0,1] \setminus M(x,Q)$. Define $r(n)$ to be the number of solutions of the equation $n = x + y$ with $x, y \in \mathcal{N}$. Then we have

$$r(n) = \int_0^1 e(-n\theta) \left(\sum_{\substack{n \leq x \\ n \in \mathcal{N}}} e(\theta n) \right)^2 d\theta = \int_{M(x,Q)} + \int_{m(x,Q)}$$

The integral over the major arcs can be evaluated whenever $\mathcal{N}$ is distributed. Thus it remains to bound the integral on the minor arcs. Hence one needs a nontrivial bound for $S(\theta) = \sum_{\substack{n \leq x \\ n \in \mathcal{N}}} e(\theta n)$ on the minor arcs. It turns out that this can be done for extremal sets. More precisely, we have the following theorem.

Theorem 1 (Brüdern). *Let $\mathcal{N}$ be a distributed set of positive density. Then the following is equivalent.*

(1) *$\mathcal{N}$ is extremal*

(2) *As Q and x tend to infinity, we have* $\int\limits_{m(x,Q)} |S(\theta)|^2 \, d\theta = o(x)$.

Hence binary additive problems involving extremal sets can be solved. E.g. the asymptotic number of representations of an integer as the sum of a k-free and an l-free number can be computed[2]. Therefore a different characterisation of extremal sets seems to be interesting. In this note we prove the following theorem.

Theorem 2. *The following two statements are equivalent:*

(1) $\mathcal{N}$ *is extremal*
(2) $\mathcal{N}$ *is $\mathcal{B}^2$-almost periodic*

Note that although additive questions involving almost periodic sets can be deatl with in an elementary way, the theory of extremal sets gives better error terms as shown in [2]. From this one obtains the following corollaries.

Corollary 3. *The intersection of extremal sets is extremal.*

This was conjectured by J. Brüdern[1] and is the real motivation of the present note. In the mean time J. Brüdern gave a different proof (personal communication).

Corollary 4. *If* $f(n) = \begin{cases} 1 & \text{if } n \in \mathcal{N} \\ 0 & \text{if } n \notin \mathcal{N} \end{cases}$ *is multiplicative, and* $\mathcal{N}$ *has positive density, then* $\mathcal{N}$ *is extremal.*

This is theorem 1.4 in [1].

We will obtain theorem 2 as a corollary of a more general statement. To formulate the next theorem, we have to introduce some notation.

We define $e_\beta(n) := e(\beta n)$. Let $1 \le q < \infty$ be a real number. On the space of functions $\mathbb{N} \to \mathbb{C}$ define a seminorm $\|f\|_q$ by $\|f\|_q^q := \limsup \frac{1}{x} \sum_{n \le x} |f(n)|^q$. Define the space $\mathcal{B}$ of periodic functions, and the space $\mathcal{A}$ of trigonometric polynomials $\sum_{\nu=1}^k a_\nu e_{\alpha_\nu}$ with α_ν real. Denote the closure of $\mathcal{B}$ with respect to $\|\cdot\|_q$ with $\mathcal{B}^q$, and the closure of $\mathcal{A}$ with $\mathcal{A}^q$. For bounded functions we have $f \in \mathcal{B}^q \Rightarrow f \in \mathcal{B}^{q'}$ for any q, q', and similar for $\mathcal{A}^q$. Define the scalar product $\langle f, g \rangle$ by

$$\langle f, g \rangle = \lim_{x \to \infty} \frac{1}{x} \sum_{n \le x} f(n)\overline{g(n)}$$

For $f, g \in \mathcal{A}^2$, this limit exists and defines a scalar product, which induces the $\|\cdot\|_2$-seminorm. Hence we can apply the theory of Hilbert spaces to obtain Fourier-series for almost periodic functions. Especially, if $f \in \mathcal{A}^2$, for every β the scalar product $\langle f, e_\beta \rangle$ exists, and we define the Fouriercoefficient of f for β to be this product. We define the spectrum of f to be the set of β, such that $\langle f, e_\beta \rangle \ne 0$ and write $\text{spec}(f)$ for this set. Now let $\alpha = (\alpha_n)$ be a sequence of numbers from the interval $[0, 1]$. Define the major arcs $M_\alpha(x, Q)$ with respect to this sequence by $M_\alpha(x, Q) = \bigcup_{n \le Q} \left[\alpha_n - \frac{Q}{x}, \alpha_n + \frac{Q}{x}\right]$ and $m_\alpha(x, Q) = [0, 1] \setminus M_\alpha(x, Q)$. A set $\mathcal{N}$ is called α-extremal if it has positive density, all Fouriercoefficients of its characteristic function exist, and we have $\int\limits_{m_\alpha(x,Q)} |S(\theta)|^2 d\theta = o(x)$ for any $Q = Q(x)$, tending to infinity with x.

Now we can state our main theorem.

Theorem 5. *Let $\mathcal{N}$ be a set of integers with positive density, f be the characteristic function of $\mathcal{N}$. Let $\alpha = (\alpha_n)$ be some sequence with $\alpha_n \in [0,1)$. Then the following statements are equivalent:*

1. *$\mathcal{N}$ is α-extremal*
2. *$f \in \mathcal{A}^2$, and $spec(f)$ is contained in α*

As theorem 1, this can be applied to additive questions.

Corollary 6. *Denote with $r(n)$ the number of positive integer solutions of the equation $n = [\sqrt{2}a] + [\sqrt{3}b]$. Then we have for $n \to \infty$, $r(n) \sim \frac{n}{\sqrt{6}}$.*

Before we begin with proofs, we recall some facts about almost periodic functions. All statements of this paragraph can be found in [3].

Lemma 7. *Let $f \in \mathcal{A}^2$ be a bounded. Then the series $\displaystyle\sum_{\beta \in spec(f)} \langle f, e_\beta \rangle e_\beta$ converges to f with respect to the $\| \cdot \|_2$-seminorm.*

Lemma 8. *The spectrum of any $f \in \mathcal{A}^2$ is countable. Furthermore, $f \in \mathcal{B}^2$ if and only if $f \in \mathcal{A}^2$, and $spec(f) \subseteq \mathbb{Q}$.*

For proofs, see [3], chapter VI.3.

Lemma 9. *If $f, g \in \mathcal{A}^2$ are bounded, we get $fg \in \mathcal{A}^2$.*

Proof: Since fg is bounded, it suffices to show that $fg \in \mathcal{A}^1$. This follows from [3], theorem VI.2.3.

Further we will use the characterization of multiplicative almost periodic functions.

Lemma 10. *Let f be a multiplicative function with mean value $M(f) \neq 0$ and $|f(n)| \leq 1$ for all n. Then $f \in \mathcal{B}^2$ if and only if the following series converge:*

1. *$S_1 = \sum_p \frac{1}{p}(f(p) - 1)$*
2. *$S_2 = \sum_p \frac{1}{p}|f(p) - 1|^2$*

Proof: This is a special case of [3], Theorem VII.5.1.

First we show that theorem 5 implies theorem 2. Assume that theorem 5 holds. We claim that all the following statements are equivalent.

1. $\mathcal{N}$ is extremal
2. $\mathcal{N}$ is α-extremal, where α is some denumeration of the rational numbers in $[0,1)$
3. $f \in \mathcal{A}^2$, and $spec(f)$ is rational
4. $f \in \mathcal{B}^2$.

The equivalence of 1. and 2. is obvious from the definition and theorem 1. Note that the existence of the Fourier-coefficients is equivalent to the fact that $\mathcal{N}$ is distributed. The equivalence of 2. and 3. is given by theorem 5, and the equivalence of 3. and 4. is given by theorem 8. Hence 1. and 4. are equivalent, which proves theorem 2.

Corollary 3 follows from theorem 2 and theorem 9, and corollary 4 follows from theorem 2 and theorem 10, where the convergence of the series is implied by the fact that f takes values in $\{0, 1\}$, hence convergence of the series is equivalent to the condition $M(f) \neq 0$.

To prove corollary 6, note that the condition "$\exists a : [\sqrt{2}a] = k$" is equivalent to the condition $[(k+1)/\sqrt{2}] - [k/\sqrt{2}] = 1$. Replacing the square brackets by approximating exponential polynomials, we see that the characteristic function f of the set $\{k | \exists a : [\sqrt{2}a] = k\}$ is in $\mathcal{A}^2$ with $\mathrm{spec} f = k/\sqrt{2} \bmod 1$, and the corresponding statement is true for the set $\{k | \exists a : [\sqrt{3}a] = k\}$. Now

$$r(n) = \int_0^1 e(-n\theta) S_{\sqrt{2}}(\theta) S_{\sqrt{3}}(\theta) d\theta$$

where $S_{\sqrt{2}}(\theta) = \sum_{a \leq x/\sqrt{2}} e(\theta[\sqrt{2}a])$, and $S_{\sqrt{3}}$ is defined similar. Since we are not interested in an error term, we choose Q tending to infinity with x sufficiently slowly. Now if we define $M_{\sqrt{2}}(x,Q) := \bigcup_{|q| \leq Q} \left[\left(\frac{q}{\sqrt{2}} \bmod 1 \right) - \frac{Q}{x}, \left(\frac{q}{\sqrt{2}} \bmod 1 \right) + \frac{Q}{x} \right]$, and $M_{\sqrt{3}}$ in the same way, we have for x sufficiently large $M_{\sqrt{2}}(x,Q) \cap M_{\sqrt{3}}(x,Q) = \left[\frac{-Q}{x}, \frac{Q}{x} \right]$ since $\sqrt{2}$ and $\sqrt{3}$ are linear independent over the rationals. This interval contributes $\frac{n}{\sqrt{6}} + o(n)$ to the whole integral, hence it suffices to estimate the remaining arcs. We have

$$\int_{-\frac{Q}{x}}^{1-\frac{Q}{x}} |S_{\sqrt{2}}(\theta) S_{\sqrt{3}}(\theta)| \, d\theta \quad \leq \quad \int_{m_{\sqrt{2}}(x,Q)} |S_{\sqrt{2}}(\theta) S_{\sqrt{3}}(\theta)| d\theta$$

$$+ \int_{m_{\sqrt{3}}(x,Q)} |S_{\sqrt{2}}(\theta) S_{\sqrt{3}}(\theta)| d\theta$$

$$\leq \quad \left(\int_{m_{\sqrt{2}}(n,Q)} |S_{\sqrt{2}}(\theta)|^2 \, d\theta \cdot \int_0^1 |S_{\sqrt{3}}(\theta)|^2 \, d\theta \right)^{1/2}$$

$$+ \left(\int_0^1 |S_{\sqrt{2}}(\theta)|^2 \, d\theta \cdot \int_{m_{\sqrt{3}}(x,Q)} S_{\sqrt{3}}(\theta)|^2 \, d\theta \right)^{1/2}$$

The second integral is $< n$, and the first integral is $o(n)$, by the definition of α-extremality. Hence the first summand is $o(n)$, and the second summand can be dealt with similary.

Thus, it suffices to prove theorem 5.

Assume that $f \in \mathcal{A}^2$, and let (α_n) be an enumeration of $\mathrm{spec}(f)$. Choose $\epsilon > 0$. Then there is some N, such that $\|f - \sum_{\nu \leq Q} \langle f, e_{\alpha_\nu} \rangle e_{\alpha_\nu} \|_2^2 \leq \epsilon$. Using the orthogonality of $e(\alpha n)$ and Parsevals equation we get

$$\int_0^1 \left| \sum_{n \leq x} \left(f(n) - \sum_{\nu \leq Q} \langle f, e_{\alpha_\nu} \rangle e_{\alpha_\nu} \right) e(\theta n) \right|^2 \, d\theta \quad = \quad \|f - \sum_{\nu \leq N} \langle f, e_{\alpha_\nu} \rangle e_{\alpha_\nu} \|_2^2 x$$

$$\leq \quad \epsilon x$$

Hence to prove that $\int_{m_\alpha(x,Q)} |S(\theta)|^2 d\theta = o(x)$, it suffices to prove this with f replaced by some sufficiently long partial sum of its Fourier-series. For if $g = \sum_{\nu \leq N} a_\nu e_{\alpha_\nu}$,

and G is the corresponding exponential sum, we have

$$\int_{m_\alpha(x,Q)} |S(\theta)|^2 d\theta \leq 2 \int_{m_\alpha(x,Q)} |S(\theta) - G(\theta)|^2 d\theta + \int_{m_\alpha(x,Q)} |G(\theta)|^2 d\theta$$

The first integral is $\leq \int_0^1 |S(\theta) - G(\theta)|^2 d\theta = \sum_{n \leq x}(f(n) - g(n))^2 \leq \epsilon x$, thus it suffices to show that the second integral is small, too. Now we have

$$\int_{m_\alpha(x,Q)} \left| \left(\sum_{\nu \leq N} a_\nu e_{\alpha_\nu}(n) \right) e(\theta n) \right|^2 d\theta \leq \underbrace{\left(\sum_{\nu \leq N} |a_\nu|^2 \right)}_{\leq 1} \cdot \sum_{\nu \leq N} |a_\nu| \int_{m_\alpha(x,Q)} \left| \sum_{n \leq x} e((\theta - \alpha_\nu n)) \right|^2 d\theta$$

$$\leq 2N \int_{Q/x}^{1-Q/x} \left| \sum_{n \leq x} e(\theta n) \right|^2 d\theta$$

$$\leq \frac{4Nx}{Q}$$

Thus for any given $\epsilon > 0$ we find some $N(\epsilon)$, such that

$$\int_{m_\alpha(x,Q)} |S(\theta)|^2 d\theta \leq \epsilon x + \frac{N(\epsilon)x}{Q}$$

With $\epsilon \to 0$ and $Q \to \infty$, this becomes $o(x)$, thus $\mathcal{N}$ is α-extremal.

Now assume that $\mathcal{N}$ is an α-extremal set of integers, and let $\epsilon > 0$. By the definition of α-extremal, we get $\int_{m(x,Q)} |S(\theta)|^2 d\theta = o(x)$, where $M(x,Q) = \bigcup_{q \leq Q} \bigcup_{(a,q)=1} \left[\frac{a}{q} - \frac{\omega(x)}{x}, \frac{a}{q} + \frac{\omega(x)}{x} \right]$, $m(x,Q) = \left[\frac{\omega(x)}{x}, 1 - \frac{\omega(x)}{x} \right]$, and $\omega(x) \nearrow \infty$ will be chosen later. Choose Q such that for all $x > x_0$ we have $\int_{m_\alpha(x,Q)} |S_x(\theta)|^2 d\theta < \epsilon x$. Set $a_\nu = \lim_{x \to \infty} \frac{1}{x} S(\alpha_\nu)$. Note that this limit exists since all Fouriercoefficients of f exist. Then define $f_Q(n) = \sum_q \leq Q \sum_{\nu \leq Q} a_\nu e(-\alpha_\nu n)$. Obviously, f_Q is a trigonometric polynomial, thus it suffices to show that the sequence f_Q approximates the characteristic function of $\mathcal{N}$. Using orthogonality of $e(\alpha)$, we have

$$\sum_{n \leq x} |f(n) - f_Q(n)|^2 = \int_0^1 |\sum_{n \leq x}(f(n) - f_Q(n))e(\theta n)|^2 d\theta$$

$$\leq \int_{M_\alpha(x,Q)} |\sum_{n \leq x}(f(n) - f_Q(n))e(\theta n)|^2 d\theta$$

$$+ \int_{m_\alpha(x,Q)} |\sum_{n \leq x} f(n)e(\theta n)|^2 d\theta + \int_{m_\alpha(x,Q)} |\sum_{n \leq x} f_Q(n)e(\theta n)|^2 d\theta$$

$$= \int_1 + \int_2 + \int_3$$

The estimation of $\int_2$ and $\int_3$ is straightforward: By assumption, $\int_2 \leq \epsilon x$, and the inequality $\int_3 \ll \frac{x}{\omega(x)}$ can be deduced as above. Thus it suffices to consider $\int_1$. Since there are no more than Q^2 major arcs, it suffices to show that the integral taken over a single major arc is $o(x)$. Hence we have to show that

$$\int\limits_{\alpha_\nu-\omega(x)/x}^{\alpha_\nu+\omega(x)/x} |\sum_{n\leq x}(f(n)-f_Q(n))e(\theta n)|^2 d\theta = o(x)$$

Using partial summation the integral becomes

$$\int\limits_{-\omega(x)/x}^{\omega(x)/x} \left| \sum_{n\leq x}(f(n)-f_Q(n))e(\alpha_\nu n) - \sum_{n\leq x}\sum_{k\leq n}(f(k)-f_Q(k))e(k\alpha_\nu)(e(\theta(n+1))-e(\theta n)) \right|^2 d\theta$$

Since $\sum_{n\leq x} f(n)e(\alpha_\nu n) \sim a_\nu x$ and $\sum_{n\leq x} f_Q(n)e(\alpha_\nu n) = a_\nu x + O(1)$, there is some function $\phi(x)$, tending monotonically to ∞, such that $\sum_{n\leq x}(f(n)-f_Q(n))e(\alpha_\nu n) < \frac{x}{\phi(x)}$. Using this function, the integral can be estimated by

$$\int\limits_{\alpha_\nu-\omega(x)/x}^{\alpha_\nu+\omega(x)/x} \left| \frac{x}{\phi(x)} + \sum_{n\leq x}\frac{n}{\phi(n)}\theta \right|^2 d\theta < \frac{2x\omega(x)^2}{\phi(x)^2} + \frac{2x\omega(x)^3}{\phi(\sqrt{x})^2} + \omega^3(x)$$

Putting the estimates for $\int_i$ together, and summing over all the $\ll Q^2$ major arcs, we get

$$\sum_{n\leq x}|f(n)-f_Q(n)|^2 < \epsilon x + c\frac{x}{\omega(x)} + \frac{4xQ^2\omega(x)^3}{\phi(\sqrt{x})^2} + \omega^3(x)Q^2$$

With $\omega(x) = \min(\phi(\sqrt{x})^{1/2}Q^{-2}, x^{1/4}Q^{-2})$, the right hand side becomes $(\epsilon+o(1))x$, since this is true for any $\epsilon > 0$, it is $o(x)$. Hence $f_Q \to f$ with respect to the $\mathcal{A}^2$-norm, i.e. f is $\mathcal{A}$-almost periodic, and $\mathrm{spec} f$ is contained in α.

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