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Analysis

Uniformly–almost–even Functions with prescribed Values, IV. Application of Gelfand's Theory

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Dedicated to LUTZ LUCHT
on the occasion of his 60^{th} birthday

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Abstract.

Given integers $0 < a_1 < a_2 < \ldots$ *and bounded complex numbers* $b_1, b_2, \ldots,$ *we deal with the problem of the existence of a uniformly–almost–even function* f *satisfying*

$$f(a_n) = b_n, \quad \textit{for all} \quad n \in \mathbb{N}.$$

In [9] *this problem was solved using elementary arguments. Now we use Gelfand's theory of commutative Banach–algebras to give sufficient conditions that there exists a function* f *with this interpolation property.*

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1 Introduction

A function $f : \mathbb{N} \to \mathbb{C}$ is called *r–even*, if the equation $f(n) = f\big(\gcd(n, r)\big)$ holds for all integers n: f is called *even*, abbreviated $f \in \mathcal{B}$, if there is some r for which f is r–even.

The closure of $\mathcal{B}$ with respect to the "*uniform*" norm $\|f\|_u = \sup_{n\in\mathbb{N}} |f(n)|$ is the complex algebra $\mathcal{B}^u$ of *uniformly-almost-even functions*. Starting with the complex vector space $\mathcal{D}$ of all *periodic* arithmetical functions, which is generated by the functions $\{n \mapsto \exp(2\pi i \frac{k}{r} \cdot n), \gcd(k,r) = 1\}$ one obtains similarly the algebra $\mathcal{D}^u$ of *uniformly-almost-periodic functions* (see, for example, [7], IV.1).

As in [3], [8], and [9], in this note the following *interpolation problem* is dealt with: Let $\{a_n\}_n$ be a strictly increasing sequence of positive integers, and $\{b_n\}_n$ a bounded sequence of complex numbers; does a uniformly-almost-even function f (resp. a uniformly-almost-periodic function) exist with values

$$f(a_n) = b_n \quad \text{for } n = 1, 2, \ldots \quad ?$$

In [9], this problem was solved, using a complicated elementary method. In this paper it is shown, that GELFANDs theory of commutative Banach-algebras, which was used already in [3], gives a simpler solution of the problem stated.[1]

Notations. $\mathbb{N} = \{1, 2, \ldots\}$ is the set of positive integers, $\mathbb{P} = \{2, 3, 5, \ldots\}$ the set of primes. For $n \in \mathbb{N}$, $p \in \mathbb{P}$, we denote by $o_p(n)$ the order of p in the factorization of n, so that $p^{o_p(n)} \mid n$, but $p^{o_p(n)+1} \nmid n$.

2 Results

Theorem 1.

Let a strictly increasing sequence $\{a_n\}_{n\in\mathbb{N}}$ of positive integers and a bounded sequence $\{b_n\}_{n\in\mathbb{N}}$ of complex numbers be given with the following property:

If $\{n_k\}_{k\in\mathbb{N}}$ is any strictly increasing sequence of positive integers such that for any $r \in \mathbb{N}$ the sequence $\{\gcd(a_{n_k}, r!)\}_{k\in\mathbb{N}}$ is eventually constant, then the limit

$$\lim_{k\to\infty} b_{n_k} \quad \textit{exists,}$$

and, in the case that, with some integer m [not depending on r],

$$\lim_{k\to\infty} \gcd(a_{n_k}, r!) = \gcd(a_m, r!)$$

for every r, its value is b_m.

Then there is a function $f \in \mathcal{B}^u$ with values $f(a_n) = b_n$ for all $n \in \mathbb{N}$.

In [9] it was shown that Theorem 1 has the following Corollaries.

Corollary 1.1.

If $\{a_n\}_n$ is a strictly monotone sequence of positive integers > 1, with the property that the minimal prime divisor $p_{\min}(a_n)$ of a_n tends to infinity, and if $\{b_n\}_n$ is a

[1]In particular, it is seen that the conditions of Theorems 1 and 2 are "natural" ones to ensure continuity of the functions F and G (see sections 4, 5).

convergent sequence in $\mathbb{C}$, then there is a function $f \in \mathcal{B}^u$ assuming the values $f(a_n) = b_n$ for $n = 1, 2, \ldots$.[2]

Corollary 1.2.

If $\{a_n\}_n$ is a strictly monotone sequence of positive integers, so that $a_m \nmid a_n$ for any m less than n, and if $\{b_n\}_n$ is a convergent sequence in $\mathbb{C}$, then there is a function $f \in \mathcal{B}^u$ assuming the values $f(a_n) = b_n$ for $n = 1, 2, \ldots$.

Corollary 1.3.

If $\{a_n\}_n$ is a strictly monotone sequence of positive integers, so that $a_m \mid a_n$ for any m less than n, and if $\{b_n\}_n$ is a convergent sequence in $\mathbb{C}$, then there is a function $f \in \mathcal{B}^u$ assuming the values $f(a_n) = b_n$ for $n = 1, 2, \ldots$.

The interpolation problem in $\mathcal{D}^u$ is dealt with in the next theorem.

Theorem 2.

Let $\{a_n\}_n$ be a strictly increasing sequence of positive integers and $\{b_n\}_n$ a bounded sequence of complex numbers.
Assume that the sequence $\{b_{n_k}\}_k$ is convergent for any strictly increasing sequence $\{n_k\}_k \in \mathbb{N}$ with the property that

for every $q \in \mathbb{N}$ there is a $k_q \in \mathbb{N}$ so that $a_{n_k} \equiv a_{n_{k'}} \mod q$ for all $k, k' > k_q$, and that $\lim_{k\to\infty} b_{n_k} = b_m$, if for any $q \in \mathbb{N}$ there are integers k_q, m so that $a_{n_k} \equiv a_m \mod q$ for all $k > k_q$.

Then there is a function $f \in \mathcal{D}^u$ with values $f(a_n) = b_n$.

Corollary 2.1.

If $\{a_n\}_n$ is strictly increasing and $\{b_n\}_n$ is convergent, then there is a function $f \in \mathcal{D}^u$ assuming the values $f(a_n) = b_n$, if in the case $\lim_{n\to\infty} b_n = b_m$ for some m, the relation $a_n \equiv a_m \mod q$ holds for any q and all sufficiently large integers n.

3 Gelfand's Theory, Tietze's Extension Theorem

For the sake of completeness we state some facts from GELFANDs Theory (see [4], **18**, [5], p. 268ff). For a commutative Banach–algebra $\mathcal{A}$ (with unit element e and with norm $\|\cdot\|$) denote by

$$\Delta_{\mathcal{A}} = \{h : \mathcal{A} \to \mathbb{C},\ h \text{ is a Banach–algebra–homomorphism}\}$$

the set of algebra–homomorphisms defined on $\mathcal{A}$. Any $h \in \Delta_{\mathcal{A}}$ is continuous, and any maximal ideal in $\Delta_{\mathcal{A}}$ is the kernel of some $h \in \Delta_{\mathcal{A}}$. The Gelfand–transform $\hat{x}$ of $x \in \mathcal{A}$ is

$$\hat{x} : \Delta_{\mathcal{A}} \to \mathbb{C},\ \hat{x}(h) \stackrel{def}{=} h(x),$$

[2] In [6] a simple elementary proof was attempted. However, unfortunately there is a gap in the proof.

and so $\hat{}$ is a map

$$\hat{} : \mathcal{A} \to \hat{\mathcal{A}} = \{\hat{x} : \Delta_{\mathcal{A}} \to \mathbb{C}.\ x \in \mathcal{A}\}.$$

Under the weakest topology. which makes every $\hat{h}$ continuous. $\Delta_{\mathcal{A}}$ becomes a compact topological Hausdorff space.
If $\mathcal{A}$ is a semi-simple[3] B^*-algebra.[4] then the Gelfand-transform $\hat{}$ is an isometric isomorphism of $\mathcal{A}$ onto $C(\Delta_{\mathcal{A}})$. the algebra of complex-valued continuous functions on $\Delta_{\mathcal{A}}$ with the sup-norm.

In sections 4 resp. 5. the GELFAND theory will be applied to the commutative Banach algebras $\mathcal{B}^u$ resp. $\mathcal{D}^u$: these algebras are semi-simple and have an involution (namely complex conjugation).

3.1 The Maximal Ideal Space of $\mathcal{B}^u$

All the homomorphisms h from the "maximal ideal space" $\Delta_{\mathcal{B}}$ of $\mathcal{B}^u$ are given (see for example [3] or [7], Chapter 4) as follows:
For any vector $\mathcal{K} = (e_p)_{p\in\mathbb{P}}$. where e_p is an integer from $[0. \infty[$ or equal to ∞. and any function $f \in \mathcal{B}^u$. define a "function value"

$$f(\mathcal{K}) = \lim_{r\to\infty} f\left(\prod_{p\le r} p^{\min\{r.e_p\}}\right).$$

For $f \in \mathcal{B}^u$. this limit does exist. If $\mathcal{K}$ has only finitely many entries $e_p \neq 0$. and if none of these is equal to ∞. then

$$f(\mathcal{K}) = f\left(\prod_p p^{e_p}\right).$$

Define

$$h_{\mathcal{K}} : \Delta_{\mathcal{B}} \to \mathbb{C} \text{ by } h_{\mathcal{K}}(f) = f(\mathcal{K}).$$

Then the maximal ideal space $\mathcal{B}^u$ of $\mathcal{B}$ is[5,6] the set of all $h_{\mathcal{K}}$. where $\mathcal{K} = (e_p)_{p\in\mathbb{P}}$. If $n = \prod_p p^{o_p(n)}$ is an integer, then the evaluation-homomorphismus $h_n : f \mapsto f(n)$ equals $h_{\mathcal{K}_n}$, where $\mathcal{K}_n = \{o_p(n).\ p \in \mathbb{P}\}$.
A subbasis of the topology on $\Delta_{\mathcal{B}}$ is given by the vectors $(*.\dots,*,e_p,*,*.\dots)$, where e_p is fixed and finite. or $e_p \geq$ some constant, and $*$ are arbitrary elements of $[0. \infty]$.

[3]The radical of $\mathcal{A}$, which is the intersection of all maximal ideals. equals (0).
[4]there is an involution $^*: \mathcal{A} \to \mathcal{A}$ satisfying $||x \cdot x^*|| = ||x||^2$.
[5]see [3].
[6]$\Delta_{\mathcal{B}}$ can also be described as the topological product

$$\prod_p \{1, p^1, p^2, \dots, p^\infty\},$$

where $\{1, p^1, p^2, \dots, p^\infty\}$ is the one-point-compactification of the discrete space $\{1. p^1, p^2 \dots\}$.

3.2 The Maximal Ideal Space of $\mathcal{D}^u$

Define X to be the compact topological product of the discrete residue class rings $\mathbb{Z}/(r\cdot\mathbb{Z})$.

$$X = \prod_{r\in\mathbb{N}} \mathbb{Z}/(r\cdot\mathbb{Z}).$$

For $m \mid n$ define the projection $\pi_{m,n}$

$$\pi_{m,n} : \mathbb{Z}/(n\cdot\mathbb{Z}) \to \mathbb{Z}/(m\cdot\mathbb{Z} =: \quad (a \bmod n) \mapsto (a \bmod m).$$

For $d \mid m$ and $m \mid n$ the relation $\pi_{d,n} = \pi_{d,m} \circ \pi_{m,n}$ holds.

$$\begin{array}{cc} \mathbb{Z}/n\cdot\mathbb{Z} & a \bmod n \\ \downarrow \pi_{m,n} & \downarrow \\ \mathbb{Z}/m\cdot\mathbb{Z} & a \bmod m \end{array}$$

The map $\pi_{m,n}$ (for $m \mid n$)

The maximal ideal space $\Delta_{\mathcal{D}}$ of $\mathcal{D}^u$ is the Prüfer ring $\hat{\mathbb{Z}}$, where

$$\hat{\mathbb{Z}} = \{\{\alpha_n\}_{n\in\mathbb{N}} \in X,\ \alpha_n \in \mathbb{Z}/n\cdot\mathbb{Z} \text{ and } \pi_{m,n}(\alpha_n) = \alpha_m, \text{ if } m \mid n\}.$$

Then $\Delta_{\mathcal{D}}$ is homeomorphic to $\hat{\mathbb{Z}}$. Denote by φ this homeomorphism $\varphi : \Delta_{\mathcal{D}} \to \hat{\mathbb{Z}}$. The evaluation homomorphisms $h_a : f \mapsto f(a)$ (for $a \in \mathbb{N}$) are dense in $\Delta_{\mathcal{D}}$.[7] In [7], p. 148, it is described how to construct the image $\varphi(h)$ for a given homomorphism $h \in \mathcal{D}^u$. It follows that

$$\varphi(h_a) = \big(a \bmod r\big)_{r\in\mathbb{N}} \text{ for an evaluation homomorphism } h_a.$$

If $\{\alpha_r\}_r$ is given, then an algebra homomorphism $h \in \Delta_{\mathcal{D}}$ mapped by φ to $\{\alpha_r\}_r$ is constructed as follows:
Define $h : \mathcal{D} \to \mathbb{C}$ on the basis elements $n \mapsto \exp(2\pi i n \cdot \frac{k}{r})$ (where $\gcd(k,r) = 1$) by

$$h\left((n \mapsto \exp(2\pi i n \cdot \frac{k}{r})\right) = \exp\left(2\pi i \cdot \frac{k}{r} \cdot \alpha_r\right).$$

and extend it linearly to $\mathcal{D}$ and then continuously to $\mathcal{D}^u$.
Write $\varphi(h) = \{\alpha_r\}_r$, and $\varphi(h') = \{\beta_r\}_r$; then the homomorphisms h and $h' \in \Delta_{\mathcal{D}}$ are "near" if and only if $\{\alpha_r\}_r$ and $\{\beta_r\}_r$ are "near", and this is true if and only if $\alpha_r \equiv \beta_r \bmod r$ for $1 \le r \le R$.

3.3 Tietze's Theorem

TIETZEs extension theorem states:[8]

> *If Y is a non-void compact subset of the locally compact Hausdorff space X, and if $f : Y \to \mathbb{C}$ is a continuous map, then there is a continuous function $F : X \to \mathbb{C}$ with compact support, extending f (so that $F|_Y = f$).*

[7] see, for example, [7], p. 148ff.
[8] see, for example, [1].

4 Proof of Theorem 1.

Define the subset $\mathcal{E} = \mathcal{E}(a_n)$ of $\Delta_{\mathcal{B}}$ as the countable [discrete] set of evaluation homomorphisms

$$\mathcal{E} = \{h_{a_n},\ n = 1, 2, \ldots\}.$$

Denote by $\mathcal{H}$ the set of accumulation points of $\mathcal{E}$. The union

$$K = \mathcal{E} \cup \mathcal{H} \subset \Delta_{\mathcal{B}}$$

is closed,[9] and therefore compact.
Define a function $F : K \to \mathbb{C}$,

firstly for points $h_{a_n} \in \mathcal{E}$ by

$$F(h_{a_n}) = b_n,$$

next for points $\eta = h_{\mathcal{K}} \in \mathcal{H}$ as follows: choose a sequence $\left\{h_{a_{n_k}}\right\}_k$ converging to η, and define

$$F(h_{\mathcal{K}}) = \lim_{k\to\infty} b_{n_k}.$$

This limit exists, because for any r the sequence $\{\gcd(a_{n_k}, r!)\}_{k\in\mathbb{N}}$ is eventually constant:

Write

$$\eta = h_{\mathcal{K}}, \text{ where } \mathcal{K} = \{e_p,\ p \in \mathbb{P}\}.$$

If e_p is finite, then $e_p = \lim_{k\to\infty} \mathrm{o}_p(a_{n_k})$, and so $\mathrm{o}_p(a_{n_k})$ is eventually constant.
If $e_p = \infty$ then $\mathrm{o}_p(a_{n_k}) \to \infty$, and so $\{\gcd(a_{n_k}, r!)\}_{k\in\mathbb{N}}$ is eventually constant.

The function F is well-defined.

Assume that $\{a_{n_k}\}_k \to \eta$, and that $\{a_{j_\ell}\}_\ell \to \eta$. Then the "union-sequence" $a_{n_1}, a_{j_1}, a_{n_2}, a_{j_2}, \ldots$ also tends to η, therefore the corresponding sequence of the b-s is convergent (due to our assumption), and the partial sequences $\{b_{n_k}\}_k$ and $\{b_{j_\ell}\}_\ell$ tend to the same limit.

Finally, F is continuous on K.

Consider a point $\eta \in \mathcal{H}$. There is a sequence $\left\{h_{a_{n_k}}\right\}_k$ converging to η. If $\eta \notin \mathcal{E}$, then $F(\eta) = \lim_{k\to\infty} b_{n_k}$, and F is continuous at the point η.

If $\eta \in \mathcal{E}$, say, $\eta = h_{a_m}$, where $a_m = \prod_{\ell=1}^{L} p_\ell^{\mathrm{o}_{p_\ell}(a_m)}$, then, for sufficiently large k, $h_{a_{n_k}} = h_{\mathcal{K}_k}$, where $\mathcal{K}_k$ is of the form

$$(\mathrm{o}_2(a_m), \mathrm{o}_3(a_m), \ldots, \mathrm{o}_{p_L}(a_m), 0, 0, \ldots, 0, *, *, \ldots).$$

[9]see, for example, [2].

Therefore, for every $r \in \mathbb{N}$,

$$\gcd(a_m, r!) = \lim_{k\to\infty} \gcd(a_{n_k}, r!),$$

and so – due to our assumptions –

$$\lim_{k\to\infty} b_{n_k} = b_m,$$

and F is continuous in h_{a_m}.

Therefore, by the TIETZE extension theorem there is a continuous function $F^* : \Delta_{\mathcal{B}} \to \mathbb{C}$, extending F. By GELFANDs theory, F^* is the image of some function $f \in \mathcal{B}^u$, $F^* = \hat{f}$, and due to

$$f(a_n) = h_{a_n}(f) = \hat{f}(h_{a_n}) = F^*(h_{a_n}) = F(h_{a_n}) = b_n$$

the function f solves the interpolation problem $f(a_n) = b_n$.

In [9] the Corollaries were deduced from Theorem 1. Using GELFANDs theory, one uses the set $\mathcal{E}$ as above. In the case of Corollary 1.1, $\mathcal{H} = h_1$ due to the condition $p_{\min}(a_n) \to \infty$, and F, defined by $F(h_{a_n}) = b_n$, $F(h_1) = \lim_{n\to\infty} b_n$ gives a continuous function.
In the case of Corollary 1.3, the condition $a_m \mid a_n$ for all $m < n$ implies that $o_p(a_n)$ is monotonely increasing, so $\lim_{n\to\infty} o_p(a_n) = e_p$ exists (possibly $e_p = \infty$). Then $\lim_{n\to\infty} h_{a_n} = h_{\mathcal{K}}$, where $\mathcal{K} = \{e_p, p \in \mathbb{P}\}$, and the definition $F(h_{\mathcal{K}}) = \lim_{n\to\infty} b_n$ makes F continuous on K.
For Corollary 1.2, the definition $F(\eta) = \lim_{n\to\infty} b_n$ for every point of accumulation η of $\mathcal{E}$ makes F continuous on K.

5 Proof of Theorem 2.

Given sequences $\{a_n\}_n$ and $\{b_n\}_n$ with the properties stated in Theorem 2 in section 2, we define the set

$$\mathcal{E} = \{h_{a_n}, n \in \mathbb{N}\}$$

and the set $\mathcal{H}$ of its points of accumulation. The set

$$K = \mathcal{E} \cup \mathcal{H} \subset \Delta_{\mathcal{D}}$$

is closed and therefore compact. Define, as in section 4, a function $G : K \to \mathbb{C}$ by

$$G(h_{a_n}) = b_n \text{ on evaluation homomorphisms } h_{a_n},$$

and

$$G(\eta) = \lim_{k\to\infty} b_{n_k}, \text{ if } \lim_{k\to\infty} h_{a_{n_k}} = \eta.$$

This limit exists.

We have to show that for every q there is a k_q so that $a_{n_k} \equiv a_{n_\ell} \bmod q$ for any $k, \ell > k_q$. If k, ℓ are large, then $h_{a_{n_k}}$ and $h_{a_{n_\ell}}$ are near. This implies that the elements $(a_{n_k} \bmod 1, a_{n_k} \bmod 2, a_{n_k} \bmod 3, \dots)$ and $(a_{n_\ell} \bmod 1, a_{n_\ell} \bmod 2, a_{n_\ell} \bmod 3, \dots)$ of $\hat{Z}$ are near, therefore

$$(a_{n_k} \bmod r) = (a_{n_\ell} \bmod r) \text{ for } 1 \le r \le R.$$

The function G is well defined, and it is continuous on K.

If $\eta \in \mathcal{H}$, then G is continuous at the point η by its very definition. If $\eta \in \mathcal{E}$, the same argument as in the proof of Theorem 1 does apply.

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