# Finiteness of a class of Rabinowitsch polynomials

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**Abstract**

We prove that there are only finitely many positive integers $m$ such that there is some integer $t$ such that $|n^2 + n - m|$ is 1 or a prime for all $n \in [t+1, t+\sqrt{m}]$, thus solving a problem of Byeon and Stark.

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In 1913, G. Rabinowitsch[4] proved that for any positive integer $m$ with squarefree $4m-1$, the class number of $\mathbb{Q}(\sqrt{1-4m})$ is 1 if and only if $n^2+n+m$ is prime for all integers $0 \leq n \leq m-3$. Recently, D. Byeon and H. M. Stark [1] proved an analogue statement for real quadratic fields. The polynomial $f_m(x) = x^2+x-m$ is called a Rabinowitsch polynomial, if there is some integer $t$ such that $|f_m(n)|$ is 1 or a prime for all integral $n \in [t+1, t+\sqrt{m}]$. They proved the following theorem:

**Theorem 1**

1. *If $f_m$ is Rabinowitsch, then one of the following equations hold: $m = 1, m = 2, m = p^2$ for some odd prime $p$, $m = t^2 + t \pm 1$, or $m = t^2 + t \pm \frac{2t+1}{3}$, where $\frac{2t+1}{3}$ is an odd prime.*
2. *If $f_m$ is Rabinowitsch, then $\mathbb{Q}(\sqrt{4m+1})$ has class number 1.*
3. *There are only finitely many $m$ such that $4m+1$ is squarefree and that $f_m$ is Rabinowitsch.*

They asked whether the finiteness of $m$ holds without the assumption on $4m+1$. It is the aim of this note to show that this is indeed the case.

**Theorem 2** *There are only finitely many $m \geq 0$ such that $f_m$ is Rabinowitsch.*

For the proof write $4m+1 = u^2D$ with $D$ sqarefree and $u$ a positive integer. We distinguish three cases, namely $D = 1$, $1 < D < m^{1/12}$ and $D \geq m^{1/12}$, and formulate each as a seperate lemma. The first two cases are solved elementary, while the last one requires a slight extension of the argument in the case $4m+1$ squarefree given by Byeon and Stark.

**Lemma 1** *If $f_m$ is Rabinowitsch and $D = 1$, then $m = 2$.*

*Proof:* We only deal with the case $m = t^2+t+\frac{2t+1}{3}$, the other cases are similar. Assume that $D = 1$, that is $4t^2 + \frac{20t}{3} + \frac{7}{3} = u^2$. We have

$$4t^2 + 4t + 1 < 4t^2 + \frac{20t}{3} + \frac{7}{3} < 4t^2 + 8t + 4$$

that is, $2t+1 < u < 2t+2$, which is impossible for integral $t$ and $u$.

**Lemma 2** *There are only finitely many $m$ such that $f_m$ is Rabinowitsch and $1 < D < m^{1/12}$.*

*Proof:* Let $p$ be the least prime with $p \equiv 1 \pmod{4D}$ and $(p, m) = 1$. By Linnik's theorem, we have $p < D^C$ for some absolute constant $C$, moreover, for $D$ sufficiently large we may take $C = 5.5$, as shown by D. R. Heath-Brown [3]. Hence, there is some constant $D_0$ such that for $D > D_0$ we have $p < m^{1/2}/6$. By construction of $p$, in any interval of length $p$ there is some $n$ such that $x - \frac{1+u\sqrt{D}}{2}$ is not coprime to $p$, i.e. such that $p$ divides $n^2 + n - m$. If $f_m$ is Rabinowitsch, this implies $f_m(n) = \pm p$, since $f_m$ is of degree 2, this cannot happen but for 4 values of $n$. However, since $p < m^{1/2}/6$, in every interval of length $m^{1/2}$, there are at least five such values of $n$, hence, $f_m$ is not Rabinowitsch.

Finally we choose a prime number $p_D \equiv 1 \pmod{4D}$ for each $D \leq D_0$, and for $m > 6 \max p_D$ we argue as above.

**Lemma 3** *There are only finitely many $m$ such that $f_m$ is Rabinowitch and that $D \geq m^{1/12}$.*

*Proof:* We may neglect the case $m = 2$. In each of the other cases, there exists a unit $\epsilon_m$ in $\mathbb{Q}(\sqrt{D})$ with $1 < |\epsilon_m| \ll m$, more precisely, such a unit is given by

$$\begin{array}{lcll} m = t^2 & : & \epsilon_m & = 2t + \sqrt{4m+1} \\ m = t^2 + t \pm 1 & : & \epsilon_m & = \frac{2t+1+\sqrt{4m+1}}{2} \\ m = t^2 + t \pm \frac{2t+1}{3} & : & \epsilon_m & = \frac{6t+3\pm 2+3\sqrt{4m+1}}{2} \end{array}$$

Let $\epsilon_D > 1$ be the fundamental unit of $\mathbb{Q}(\sqrt{D})$. Since the group of positive units in $\mathbb{Q}(\sqrt{D})$ is free abelian of rank 1, there is some $k$ such that $\epsilon_m = \epsilon_D^k$, hence we have $\epsilon_D < m$. By the Siegel-Brauer-theorem we have $\log(h(\mathbb{Q}(\sqrt{D})) \log |\epsilon_D|) \sim \log \sqrt{D}$. If $f_m$ is Rabinowitch, then $h(\mathbb{Q}(\sqrt{D})) = 1$, and by assumption we have

$$\log |\epsilon_D| \leq \log |\epsilon_m| < \log m \leq 12 \log D,$$

hence we obtain the inequality

$$12 \log D > D^{1/2+o(1)}$$

which can only be true for finitely many $D$. Since $m \leq D^{12}$, there are only finitely many $m$, and our claim follows.

Note that Lemma 1 and Lemma 2 are effective, while Lemma 3 depends on a bound for Siegel's zero. However, one can deduce that there is an effective constant $m_0$, such that there exists at most one $m > m_0$ such that $f_m$ is Rabinowitsch.

**Note added in proof.** In the mean time, D. Byeon and H. M. Stark[2] also obtained a proof of Theorem 1, moreover, they determined all Rabinowitsch polynomials up to at most one exception. The same result has also been obtained independently by S. Louboutin.

# References

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