THE EXPONENTIAL SUM OVER SQUAREFREE INTEGERS

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Denote by $r_\nu(N)$ the number of representations of N as the sum of ν squarefree numbers. In a series of papers Evelyn and Linfoot [3]–[7] proved that

$$r_\nu(N) = \mathfrak{S}_\nu(N)N^{\nu-1} + \mathcal{O}(N^{\nu-1-\theta(\nu)+\varepsilon}),$$

where

$$\mathfrak{S}_\nu(N) = \frac{1}{(\nu-1)!}\left(\frac{6}{\pi^2}\right)^\nu \prod_{p^2\nmid N}\left(1-\frac{1}{(1-p^2)^\nu}\right)\prod_{p^2|N}\left(1-\frac{1}{(1-p^2)^{\nu-1}}\right),$$

and

$$\theta(2)=\theta(3)=\frac{1}{3}, \qquad \theta(\nu)=\frac{1}{2}-\frac{1}{2\nu} \quad (\nu\geq 4).$$

Mirsky[10] improved the error term for $\nu \geq 3$ to $\theta(\nu) = \frac{1}{2} - \frac{1}{4\nu-2}$. Using a new approach to bound the minor arc integral develloped by Brüdern, Granville, Perelli, Vaughan and Wooley[1], Brüdern and Perelli[2] showed that $\theta = \frac{1}{2}$ for all $\nu \geq 3$, and that any further improvement would imply a quasiriemannian hypothesis. Moreover, assuming the generalized riemannian hypothesis, they proved that $\theta(3) = \frac{3}{4} + \frac{1}{14}$ and $\theta(\nu) = \frac{3}{4}$ for all $\nu \geq 4$. These result are optimal apart from the summand $\frac{1}{14}$; in personal communication Brüdern conjectured that $\theta(3) = \frac{3}{4}$ should hold true. It is the aim of this note to prove this conjecture.

Define $S(\alpha) = \sum_{n\leq N}\mu^2(n)e(\alpha n)$, and, for integers N and Q satisfying $1 \leq Q < N^{1/2}/2$, let $\mathfrak{M}(Q)$ be the union of all intervals $\{\alpha : |\alpha q - a| \leq QN^{-1}\}$, where $q \leq Q$, and $(a,q) = 1$, and set $\mathfrak{m}(Q) = [QN^{-1}, 1-QN^{-1}] \setminus \mathfrak{M}(Q)$. With these notation we will prove the following.

Theorem 1. *We have $S(\alpha) \ll N^{1+\varepsilon}Q^{-1}$ for all $\alpha \in \mathfrak{m}(Q)$, provided that $Q \leq N^{1/2}$.*

Under the restriction $Q \leq N^{3/7}$, this was proven in [2, Theorem 4]. As already remarked in [2, Sec. 5], the weakening of the assumption on Q implies the following.

Theorem 2. *Assume the generalized riemannian hypothesis. Then we have*

$$r_3(N) = \mathfrak{S}(N)N^2 + \mathcal{O}(N^{5/4+\varepsilon}).$$

By Dirichlet's theorem on diophantine approximation, for every $\alpha \in \mathfrak{m}(Q)$ there exist coprime integers a, q with $q \leq NQ^{-1}$, such that $|q\alpha - a| \leq N^{-1}Q$. By the definition of $\mathfrak{m}(Q)$, we necessarily have $q > Q$. Hence, Theorem 1 is essentially equivalent to the following.

Theorem 3. *Define $S(\alpha)$ as above, and let q be an integer satisfying $|\alpha q - a| \leq q^{-1}$. Then we have*

$$|S(\alpha)| \ll N^{1+\varepsilon} q^{-1} + N^{\varepsilon} q.$$

We approach Theorem 3 by the following lemma, which replaces Lemma 1 in [2].

Lemma 1. *Let $\alpha \in (0, 1)$ be a real number, and assume that $|q\alpha - a| < \frac{1}{q}$. Let D be an integer, and denote by $W(D, z)$ the number of integers $d \leq D$ satisfying $\|d^2\alpha\| \leq z$. Then, for $D^2 > \frac{1}{4}q$, we have*

$$W(D, z) \ll D^2 q^{-1} + D^{1+\varepsilon} z^{1/2}.$$

Proof. Cut the interval $[1, D^2]$ into $K = [D^2 q^{-1}] + 1$ intervals of length q, where the last interval may be shorter. For $k \leq K$, let a_k be the number of integers d, such that $\|d^2\alpha\| \leq z$ and $kq \leq d^2 < (k + 1)q$. Then $\sum_{k \leq K} a_k = W(D, z)$, and by the arithmetic-quadratic mean inequality, $\sum_{k \leq K} a_k^2 \geq W(D, z)^2 K^{-1}$. Denote by $\mathcal{D}$ the set of all pairs (d_1, d_2) with the properties that $\|d_i^2\alpha\| \leq z$ and $1 \leq |d_1^2 - d_2^2| \leq q$. Then either $W(D, z) \leq 2K$, which is sufficiently small, or we can bound $|\mathcal{D}|$ from below via

$$|\mathcal{D}| \geq \sum_k \binom{a_k}{2} \gg \sum_k a_k^2 - \sum_k a_k \gg \sum_k a_k^2 \gg W(D, z)^2 K^{-1}.$$

Denote by $\mathcal{N} \subseteq [1, q]$ the set of all values of $|d_1^2 - d_2^2|$, where d_1, d_2 ranges over all pairs in $\mathcal{D}$. Then every pair in $\mathcal{D}$ gives rise to an element of $\mathcal{N}$, and the number of different pairs d_1, d_2 having the same difference $d_1^2 - d_2^2 = n$ is bounded above by the number of divisors of n, and therefore $\ll q^{\varepsilon}$. Hence, we decuce

$$W(D, z)^2 \ll |\mathcal{D}|K \ll |\mathcal{N}|K q^{\varepsilon}.$$

On the other hand, for every $n \in \mathcal{N}$, we have $\|n\alpha\| \leq \|d_1^2\alpha\| + \|d_2^2\alpha\| \leq 2z$, hence,

$$W(D, z)^2 \ll D^2 q^{\varepsilon - 1} \big|\{n \leq q : \|\alpha n\| \leq 2z\}\big| \ll D^2 q^{\varepsilon - 1}(qz + 1).$$

From this we obtain in the case $W(D, z) > 2K$, that

$$W(D, z) \ll D^{1+\varepsilon} z^{1/2} + D^{1+\varepsilon} q^{-1/2},$$

which is again of the right size, since $D > \frac{1}{2}q^{1/2}$. □

Proof of Theorem 3. Write

$$\begin{aligned} S(\alpha) &= \sum_{d\le\sqrt{N}} \mu(d) \sum_{m\le Nd^{-2}} e(\alpha d^2 m) \\ &\ll \log N \max_{1\le D\le\sqrt{N}/2} \sum_{D\le d<2D} \min\left(\frac{N}{D^2}, \|\alpha d^2\|^{-1}\right) \\ &= \log N \max_{1\le D\le\sqrt{N}/2} \Upsilon(\alpha, D), \end{aligned}$$

say. To prove Theorem 3, it suffices to show that $\Upsilon(\alpha, D) \ll N^{1+\varepsilon}Q^{-1}$ for all $D \le \sqrt{N}/2$. For $D > \frac{1}{4}q^{1/2}$, we have

$$\begin{aligned} \Upsilon(\alpha, D) &\ll \log N \max_{z>N/D^2} z^{-1}W(D,z) \\ &\ll \log N \max_{z>N/D^2} \left(z^{-1}D^2q^{-1} + D^{1+\varepsilon}z^{-1/2}\right) \\ &\ll N^{1+\varepsilon}q^{-1} + N^{1/2+\varepsilon}. \end{aligned}$$

For $D \le \frac{1}{4}q^{1/2}$, we argue as in the proof of [2, Lemma 1]. We have

$$|\alpha d^2 - ad^2/q| \le 4D^2|\alpha - a/q| \le 4D^2q^{-2} \le \frac{1}{4q},$$

and therefore

$$|\Upsilon(\alpha, D)| \le 2 \sum_{D\le d<2D} \left\|\frac{ad^2}{q}\right\| \ll q\log q \ll N^{\varepsilon}q.$$

Taking these estimates together, we find that

$$S(\alpha) \ll N^{1+\varepsilon}q^{-1} + N^{1/2+\varepsilon} + N^{\varepsilon}q,$$

and the second term is always dominated by either the first or the last one, which implies our theorem. □

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