ON FUNCTIONS THAT ARE g–ADDITIVE AND h–ADDITIVE

J.-C. PUCHTA AND J. SPILKER

Abstract. We caracterize the functions f, that are simultanously g–additive and h–additive if the bases g, h don't divide each other: f is a linear combination of the identity and functions that are periodic and constant on some subblocs.

1. Introduction and main result. Let $g \in \mathbb{Z}, \geq 2$ be fixed. Then every $n \in \mathbb{N}$ can be uniquely represented in the form

$$n =_{r\geq 0} e_r(n) g^r \quad \text{with} \quad e_r(n) \in \{0, 1, \ldots, g-1\}.$$

Definition. A function $f : \mathbb{N} \to \mathbb{C}$ is called

1. g–additive ($g \in \mathbb{Z}, \geq 2$) iff $f\left({}_{r\geq 0} e_r(n) g^r\right) =_{r\geq 0} f\big(e_r(n) g^r\big), \ \forall\, n \in \mathbb{N}_0$;
2. p–periodic ($p \in \mathbb{N}$) iff $f(n+p) = f(n), \ \forall\, n \in \mathbb{N}_0$;
3. q–constant ($q \in \mathbb{N}$) iff $f(aq+b) = f(aq), \ \forall\, a, b \in \mathbb{N}_0, b < q$.
4. If $g, h, k \in \mathbb{N}$, then
   $V_k = V_k(g;h) := \{f : \mathbb{N}_0 \to \mathbb{C} : f(0). = 0, \ f$ is $(g,h)^k$–periodic and $[g,h]^{k-1}$–constant$\}$; here (g,h) is the greatest common divisor of g, h and $[g,h]$ the least common multiple.

Remark. Every g–periodic function is g–additive ($g \geq 2$). Every g–additive function is g^k–additiv ($k \in \mathbb{N}$). Every p_1–periodic and p_2–periodic function is (p_1, p_2)–periodic. Every q_1–constant and q_2–constant function is $[q_1, q_2]$–constant. All functions of V_k are (g,h)–additive, g–additive and h–additive.

The first lemma contains simple properties of the spaces V_k.

Lemma 1. *Let $g, h \in \mathbb{N}$.*

a) *If $[g,h]^{k-1} \big| (g,h)^k$, then* $\dim_{\mathbb{C}} V_k = \frac{(g,h)^k}{[g,h]^{k-1}} - 1$.
b) $V_k = 0$ *if* $[g,h]^{k-1}(g,h)^k$.
c) *If $g \neq h$, there exists a $k_0 \in \mathbb{N}$, so that $V_k = 0$ for all $k \geq k_0$.*
d) *If $g \neq h$, then $V_{k_1} \cap_{k > k_1} V_k = 0$ for every $k_1 \in \mathbb{N}$.*

Date: March 1, 2001.
1991 *Mathematics Subject Classification.* 11A25 (11A63).
Key words and phrases. g–additive functions.

Proof. a) Every $f \in V_k$ is uniquely determined by the values $a[g,h]^{k-1},\ 1 \leq a < \frac{(g,h)^k}{[g,h]^{k-1}}$, and these values can be arbitrarily given.

b) If $f \neq 0$ is p–periodic, q–constant, $f(0) = 0$ and $m := \min\{n \in \mathbb{N} : f(n) \neq 0\}$, then $q|m, q|m+p$, hence $q|p$.

c) Choose k_0 such, that $(g,h)^{k_0} < [g,h]^{k_0-1}$.

d) Every $f \in V_{k_1} \cap_{k>k_1} V_k$ is $(g,h)^{k_1}$–periodic and has the property $f(n) = 0$ for all $0 \leq n < [g,h]^{k_1}$, so $f = 0$. □

Examples.

1) If $(g,h) = 1$, then $V_k(g;h) = 0,\ \forall\, k \geq 1$.
2) If g has a prime–factor, that does not divide h, then $V_k(g;h) = 0,\ \forall\, k \geq 2$.
3) The space $V_2(24;36)$ contains the function

$$f(n) = \begin{cases} 0 & n \equiv a \bmod 144 (0 \leq a < 72) \\ 1 & n \equiv b \bmod 144 (72 \leq b < 144). \end{cases}$$

Our main result is the following

Theorem. Let g,h be integers ≥ 2 and assume gh, hg. Then

(1) every function f, that is simultanously g–additive and h–additive, can be uniquely represented in the form
$$f = c \cdot id +_{1 \leq k \leq k_0} f_k, \quad \text{where} \quad c \in \mathbb{C}, f_k \in V_k(g;h).$$
(2) The complex vectorspace V of all these functions has dimension
$$\dim_{\mathbb{C}} V = 1 + \sum d_k,\ d_k := \frac{(g,h)^k}{[g,h]^{k-1}} - 1,$$
where we sum over the natural k with $[g,h]^{k-1}\big|(g,h)^k$.
(3) Every g– and h– additive function is already (g,h)–additive, if $(g,h) > 1$.

Remark. We suppose that $\frac{\log g}{\log h}$ is irrational. Then there exist natural l,m so that $g^l h^m,\ h^m g^l$. It follows from the theorem, that the space V of all g–additive and h–additive functions has finite dimension and every function of V is (g^l,h^m)–additiv, if $(g^l,h^m) > 1$. If k_0 is chosen so that $V_k(g^l;h^m) = 0, \forall\, k \geq k_0$, and if $f_k \in V_k(g^l;h^m)$, then $f_k(dn) = f_k(0) = 0$, where $d := (g^l,h^m)^{k_0}$, so $f(dn) = cn$ for every natural n.

a) If in addition g,h are coprime, on can choose $k_0 = 1$ and every $f \in V$ has the form $f(n) = cn$. This was proved by Uchida [2, theorem 1] by another method.
b) If f has the property $f(eg^r) = f(e),\ 0 \leq e < g,\ r \geq 0$ ("strongly g–additive"), then the theorem gives $f = 0$, so every strongly g–additive and h–additive function vanishes if $g \nmid h$ and $h \nmid g$. This is a result of Toshimitsu [1, theorem 3].

2. Lemmata. If $f : \mathbb{N}_0 \to \mathbb{C}$, we consider the difference function $\Delta f(n) := f(n) - f(n-1)$, $n \geq 1$.

Lemma 2. *If f is g–additive, then*

$$\Delta f(n+g) = \Delta f(n), \quad \textit{if} \quad n \not\equiv 0(\mathrm{mod} g).$$

Proof. Let $n \in \mathbb{N}_0$, say $n =_{r\geq 0} e_r(n)g^r$. Then if gn.

$$e_r(n) = e_r(n-1), e_r(n+g) = e_r(n-1+g), \quad r \geq 1$$
$$e_0(n) = e_0(n+g), e_0(n-1) = e_0(n-1+g)$$

and so

$$\begin{aligned}\Delta f(n) &= f\big(e_0(n)\big) - f\big(e_0(n-1)\big)\\ &= f\big(e_0(n+g)\big) - f\big(e_0(n-1+g)\big)\\ &= \Delta f(n+g).\end{aligned}$$

□

Lemma 3. *Let $g, h \in \mathbb{N}$, gh, hg and the function $F : \mathbb{N} \to \mathbb{C}$ have the properties*

$$F(n+g) = F(n), \textit{ if } \quad n \not\equiv 0(\mathrm{mod}\, g),$$
$$F(n+h) = F(n), \textit{ if } \quad n \not\equiv 0(\mathrm{mod}\, h).$$

Then there are functions $P, Q : \mathbb{N} \to \mathbb{C}$ so that

a) $F = P + Q$;
b) P *is (g,h)–periodic;*
c) $Q(n) = 0$, *if* $n \not\equiv 0 \pmod{[g,h]}$.

Proof. Let P be the (g,h)–periodic function defined by $P(n) = F(n)$, $1 \leq n \leq (g,h)$ and $Q := F - P$. We have to prove c). Given $n \not\equiv 0\big(\bmod\, [g,h]\big)$, we distinguish two cases.
1. case $(g,h)n$: choose $k, l \in \mathbb{Z}$ so that $kg + lh = (g,h)$; suppose $k > 0$. Then

$$\begin{aligned}F(n) &= F(n+kg), \text{ since } gn,\\ &= F(n+kg+lh), \text{ since } (g,h)n+kg, \text{ so } hn+kg,\\ &= F\big(n+(g,h)\big).\end{aligned}$$

Hence F has period (g,h) on the set $\mathbb{N} - (g,h)\mathbb{N}$, and agrees there with P, so c) in this case.
2. case $(g,h)|n$: Since $[g,h]n$, we have gn or hn, say gn. Choose $k \in \mathbb{N}$ so that $kg \equiv (g,h) - n(\mathrm{mod}\, h)$ and $l \in \mathbb{Z}$ so that $kg + lh = (g,h) - n$. Then

$$\begin{aligned}F(n) &= F(n+kg), \text{ since } gn\\ &= F\big((g,h) - lh\big)\\ &= F\big((g,h)\big), \text{ since } hg, \text{ so } h(g,h).\end{aligned}$$

Since P has period (g,h), we finally have

$$Q(n) = F(n) - P(n) = F\big((g,h)\big) - P\big((g,h)\big) = 0.$$

□

Lemma 4. *Assume gh, hg. Then every function f with $f\big((g,h)\big)=0$ that is g–additive and h–additive has a representation as sum of a (g,h)–periodic function p with $p(0)=0$ and a $[g,h]$–constant function q with $q(0)=0$.*

Proof. By Lemma 2 the function $F=\Delta f$ fulfills the assumptions of Lemma 3. So we get $\Delta f = P+Q$, where P is a (g,h)–periodic function and $Q(n)=0$ if $[g,h]n$. Now we define

$$p(n) := \begin{cases} 0, & n=0 \\ {}_{1\le m\le n}P(m), & n>0 \end{cases} \qquad q(n) := \begin{cases} 0, & n=0 \\ {}_{1\le m\le n}Q(m), & n>0. \end{cases}$$

These functions have the desired properties, since $f(n) ={}_{1\le m\le n}\,\Delta f(m)$ implies $f=p+q$, and because of $Q(m)=0$ if $a[g,h]<m<(a+1)[g,h], a\in\mathbb{N}_0$, the function q is $[g,h]$–constant and finally p has period (g,h): if $n\in\mathbb{N}_0$

$$\begin{aligned} p\big(n+(g,h)\big)-p(n) &={}_{1\le m\le (g,h)}\,P(m) = p\big((g,h)\big) \\ &= -q\big((g,h)\big),\ \text{since } f\big((g,h)\big)=0 \\ &= -q(0),\ \text{since } (g,h)<[g,h] \\ &= 0. \end{aligned}$$

□

3. Proof of the Theorem. (1) Uniqueness: let $c\cdot id +_{1\le k\le k_0} f_k = 0,\ c\in\mathbb{C},\ f_k\in V_k$. Every f_k is bounded, so $c=0$. Then Lemma 1, d) implies $f_k=0$.

Existence: let f be g–additive and h–additive, set $c := \frac{f((g,h))}{(g,h)}$. Lemma 4 gives $f-c\cdot id = f_1+q_1$ with $f_1\in V_1$ and a $[g,h]$–constant function q_1 with $q_1(0)=0$. Since $q_1 = f-c\cdot id - f_1$ is g^2–additive and h^2–additive, again by Lemma 4 $q_1 = f_2+q_2$, where $f_2\in V_2$ and q_2 is $[g,h]^2$–constant, $q_2(0)=0$. Iterating this, we get a representation $f = c\cdot id +_{1\le k\le k_0} f_k + q$, where $f_k\in V_k,\ q$ is $[g,h]^{k_0}$–constant. Again by Lemma 4 and $V_k=0,\ \forall\, k\ge k_0$ we have that q is $[g,h]^k$–constant for every $k\ge k_0$. So q is constant and $q=0$ since $q(0)=0$. This proves part 1 of the theorem.

(2) The formulae of the dimension follows by Lemma 1, a).

(3) All functions of V_k and the identity are (g,h)–additive, if $(g,h)>1$.
So the theorem is proved. □

Remarks 1) We can generalize the theorem to functions, that are additive with respect to several bases $g_i,\ 1\le i\le n$ if g_ig_j for every $i\ne j$. Let $V_k(g_1;g_2;\ldots;g_n)$ be the space of $(g_1,\ldots,g_n)^k$–periodic and $[g_1,\ldots,g_n]^{k-1}$–constant functions, that vanish at the place 0. Then the space of all functions, that simultanously
are g_i– additive $(1\le i\le n)$ is generated by the identity and the spaces V_k. This can

be proved by the theorem: let f be g_1-, g_2- and g_3- additive. Then $f - c \cdot id = p + q$ where p is (g_1, g_2)–periodic and q is $[g_1, g_2]$–constant. The same argument gives $f - \tilde{c}\, id = \tilde{p} + \tilde{q}$, where $\tilde{p}$ is (g_1, q_3)–periodic and $\tilde{q}$ is $[q_1, q_3]$–constant. We have $c = \tilde{c}$, since $(\tilde{c} - c)id = p - \tilde{p} + q - \tilde{q}$ is bounded and $q(n) = 0 = \tilde{q}(n)$ if $0 \le n < g_1$, hence $p(n) = \tilde{p}(n)$ in $0 \le n < g_1$. Since p and $\tilde{p}$ are g_1–periodic, they agree and are (g_1, g_2, g_3)–periodic. Since $p + q = \tilde{p} + \tilde{q}$, the functions $q, \tilde{q}$ agree and are $[q_1, q_2, q_3]$–constant. By continuing in that way, we get the general theorem.

2) For g–and h–multiplicative functions without zeros the analogue of the theorem is true. On has to use quotients instead of differences. But our method seems not to work, if the multiplicative functions have zeroes.

References

[1] T. Toshimitsu, *Strongly q–additive functions and algebraic independence*, Tokyo J. Math. 21 (1998), 107–113.

[2] Y. Uchida, *On p and q–additive functions*, Tokyo J. Math. 22 (1999), 83–97.

Jan–Christoph Puchta and Jürgen Spilker, Institute of Mathematics of the University of Freiburg, Eckerstr. 1, D–79104 Freiburg

E-mail address: jcp@arcade.mathematik.uni-freiburg.de

E-mail address: spilker@sun7.mathematik.uni-freiburg.de