Sign changes of $\pi(x, q, 1) - \pi(x, q, a)$

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Abstract

It is known, that under the assumption of the generalized Riemannian hypothesis, the function $\pi(x, q, 1) - \pi(x, q, a)$ has infinitely many sign changes. In this article we give an upper bound for the least such sign change. Similarly, assuming the Riemannian hypothesis we give a lower bound for the number of sign changes of $\pi(x) - \mathrm{li}\, x$. The implied results for the least sign change are weaker then those obtained by numerical methods, however, our method makes no use of computations of zeros of the ζ-function.

1 Introduction

The following question is known as the Shanks-Renyi-race problem: Given an integer q, and a bijection σ from the set $\{1, 2, \dots, \varphi(q)\}$ to the set of residue classes prime to q, is it true that there are arbitrary large values x, such that the inequalities

$$\pi(x, q, \sigma(1)) > \pi(x, q, \sigma(2)) > \cdots > \pi(x, q, \sigma(\varphi(q)))$$

hold true?In this form the problem is unsolved for all q with $\varphi(q) > 2$, even assuming the Generalized Riemannian Hypothesis. With π replaced by Ψ, it was solved by J. Kaczorowski [6] for $q = 5$, and the method develloped there can be used for other small modules, too. However, the problem involving π is far more difficult, and the only result obtained so far involving more then 2 residue classes was obtained by J. Kaczorowski [5], who showed that the function $\pi(x, q, 1) - \max\limits_{a \not\equiv 1 \pmod q} \pi(x, q, a)$ has infinitely many sign changes. In [12], the same was shown by a different method. In this note we use the method of [12] to give numerical bounds for the first sign change and for the number of sign changes up to a given bound. We will prove the following theorem.

Theorem 1. *Let q be a natural number, and set $q^+ = \max(q, e(1260))$. Assume that no L-series $\pmod q$ has zeros off the critical line. Let $f(q)$ be the number of solutions of the congruence $x^2 \equiv 1 \pmod q$. Then there is an $x < e_2((q^+)^{170} + e^{18f(q)})$ such that $\pi(x, q, 1) > \pi(x, q, a)$ for all $a \not\equiv 1 \pmod q$. Moreover, if $V(x)$ denotes the number of sign changes of $\pi(t, q, 1) - \max\limits_{a \not\equiv 1 \pmod q} \pi(t, q, a)$ in the range $2 \leq t \leq x$, we have*

$$V(x) > \frac{\log x}{\exp((q^+)^{170} + e^{18f(q)})} - 1.$$

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Here and in the sequel, $e_k(x)$ denotes the k-fold iterated exponential function, and $\log_k x$ the k-fold iterated logarithm. Note that the dependence on $f(q)$ is an immanent feature of the problem, however, for almost all q we have $f(q) < \log q$, thus the least sign change is of order less then $e_3(55 \log q \log_2 q)$ for almost all q.

By the same method bounds for sign changes of $\pi(x) - li\ x$ can be obtained. Our result on the first sign change is substantially weaker than those given by Skewes[14], Lehmann[7] and te Riele[13], however, these estimates involve large scale computation of zeros of Riemann's ζ-function and give no bound on the asymptotical behaviour of the number of sign changes.

Theorem 2. *Assume the Riemann hypothesis. Then there is an $x < e_3(16.7)$, such that $\pi(x) > li\ x$. If $V(x)$ denotes the number of sign changes of $\pi(x) - li\ x$, we have $V(x) > \frac{\log x}{e_2(16.7)} - 1$.*

A. E. Ingham[4] proved that $V(x) > c \log x - 1$ for some positive constant c, however, his method of proof was ineffective. Without the assumption of the Riemannian Hypothesis, slightly weaker estimates were given by J. Pintz (see [10] for an ineffective, [11] for an effective result). Moreover, J. Kaczorowski proved $V(x) > c \log x - 1$ unconditionally.

Since the proof of this theorem is easier, but shows all relevant details, we will give this first.

Throughout this note, ρ will denote nontrivial zeros of ζ or some L-series. Since we will always assume that all zeros are on the critical line, we can write $\rho = \frac{1}{2} + i\gamma$ with γ real. For a real number x, $\|x\|$ denotes the distance of x to the nearest integer. Similar, for $x \in \mathbf{R}^n$, define $\|x\|$ to be the distance of x to the nearest lattice point.

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2 Some Lemmata for Theorem 1

We begin our computations with the following statement on the vertical distribution of zeros of ζ.

Lemma 3. *Denote with $N(T)$ the number of zeros ρ of ζ with $0 \Re\ \rho < 1, 0 < \Im\ \rho < T$. Then for $T > 2$ we have $N(T) < \frac{1}{6} T \log T$ and $N(T+1) - N(T) < \log T$.*

In fact Backlund[2] gave a more precise estimate, however, this lemma will suffice for our purpose. Even better estimates are available under the Riemannian hypothesis, however, it seems difficult to make these improvements explicit, and the bounds obtained that way will not influence our final result significantly.

For this and the next section, define the functions $\Delta(t) = \sum_\gamma \frac{e^{it\gamma}}{\rho}$ and $\Delta_T(t) = \sum_{|\gamma|<T} \frac{e^{it\gamma}}{\rho}$, where both summations run over roots of ζ on the critical line.

Lemma 4. *Let $a > b > 0$ be real numbers with $a - b < \frac{1}{36}$ and $T > e^4$. Then we have*

$$\begin{aligned}\int_a^b |\Delta(t) - \Delta_T(t)|^2 dt &= \sum_{|\gamma_1|,|\gamma_2|>T} \frac{1}{(1/2+i\gamma_1)(1/2+i\gamma_2)} \frac{e^{b(\gamma_1+\gamma_2)} - e^{a(\gamma_1+\gamma_2)}}{\gamma_1+\gamma_2} \\ &< \frac{2}{9}\frac{\log^3 T}{T}\end{aligned}$$

If $\gamma_1 + \gamma_2 = 0$ then $\frac{e^{b(\gamma_1+\gamma_2)} - e^{a(\gamma_1+\gamma_2)}}{\gamma_1+\gamma_2}$ denotes its limit for $\gamma_2 \to -\gamma_1$, i.e. $b - a$.

We will also need the following statement, which depends on a pigeon-hole principle, for a proof see [12].

Lemma 5. *Let n and N be natural numbers, $\vec{\alpha} = (t_1, \dots, t_n) \in \mathbf{R}^n, \epsilon > 0$. Then there is a sequence of N real numbers $1 < s_1 < \dots < s_N < \frac{N2^n\Gamma(n/2)}{\pi^{n/2}\epsilon^n} + 1 =: M + 1$ such that for $1 \le i \le N$ we have*

$$\|s_i \cdot (t_1, \dots, t_n)\| < \epsilon$$

and $s_{i+1} \ge s_i + 1$.

Further we note that studying sign changes of π is equivalent to studying large values of Ψ, an observation which is made exploicit by the following lemma.

Lemma 6. *Let $x > e^{60}$ be a real number such that $\Psi(x) > x + 1.01\sqrt{x} - 2$. Then $\pi(x) > \operatorname{li} x$.*

Proof. The argument follows the lines of S. Lehmann[7]. Define $\Pi(x) = \sum_{n \le x} \frac{\Lambda(n)}{\log n}$, and $\Delta^*(x) = \Psi(x) - x$. We have an explicit formula

$$\Pi(x) = \operatorname{li} x - \sum_\rho \operatorname{li} x^\rho + \theta x^{1/3},$$

where θ is some real number, which depends on x and satisfies $|\theta| \le 1$, provided that $x > e^{12}$. Further we have

$$\operatorname{li} x^\rho = \frac{x^\rho}{\rho \log x} + \theta \frac{x^{1/2}}{|\rho|^2 \log^2 x}$$

where θ is some complex number satisfying $|\theta| \le 1$. Thus, comparing the sum over zeros with the sum occuring in the explicit formula for $\Psi(x)$, we get

$$\Pi(x) - \operatorname{li} x = \frac{\Psi(x) - x}{\log x} + \theta \left(\frac{1}{\log x} \sum_\rho \frac{1}{|\rho|^2} + x^{-1/6} \log x \right) \frac{\sqrt{x}}{\log x}.$$

Finally, again under the assumption $x > e^{12}$, we have

$$\pi(x) - \Pi(x) = -\frac{1}{2} \operatorname{li} \sqrt{x} + \theta x^{1/3}.$$

Putting these estimates together, we get

$$\pi(x) - \operatorname{li} x = \frac{\Psi(x) - x}{\log x} - \frac{\sqrt{x}}{\log x}\theta\left(\frac{1}{\log x}\sum_\rho \frac{1}{|\rho|^2} + x^{-1/6}\log x\right)\frac{\sqrt{x}}{\log x}.$$

Hence, under the assumptions $x > e^{12}$ and $\Delta^*(x) > 1.01\sqrt{x} - 2$, we get

$$\pi(x) - \operatorname{li} x \geq \frac{0.01\sqrt{x}}{\log x} - \frac{0.05\sqrt{x}}{\log^2 x} - 2x^{1/3} - 2,$$

where we used the bound $\sum_\rho \frac{1}{|\rho|^2} < 0.05$ (see the proof of the next lemma). For $x > e^{60}$, the right-hand side of the last equation becomes positive, and the proof of the lemma is complete. □

Finally we need the following quantitative version of [12], Lemma 8.

Lemma 7. *We have*

$$|\Delta(t) + \Delta(-t)| < 0.0462$$

Proof. We have

$$\begin{aligned} |g(t) + g(-t)| &= \frac{1}{2}\left|\sum_\rho \frac{e^{it\gamma} + e^{-it\gamma}}{\rho} + \frac{e^{it\gamma} + e^{-it\gamma}}{\bar{\rho}}\right| \\ &= \frac{1}{2}\left|\sum_\rho \frac{e^{it\gamma} + e^{-it\gamma}}{|\rho|^2}\right| \\ &\leq \sum_\rho \frac{1}{|\rho|^2} \\ &= 2 + C - \log\pi - 2\log 2 \\ &= 0.04619\ldots \end{aligned}$$

Here $C = 0.5772\ldots$ denotes Euler's constant. The evaluation of the sum $\sum_\rho \frac{1}{|\rho|^2}$ is given e.g. in [3]. Note that here we have twice the value given in [3], since we take the sum over all zeros, not only zeros with positive imaginary part. □

3 Proof of Theorem 2

Obviously, the lower bound for $V(x)$ implies the bound for the first sign change, hence, we will only consider the second claim of Theorem 2. Define $\Delta(t)$ and $\Delta_T(t)$ as above. We have for $t > 0$

$$\Psi(e^t) = e^t - e^{t/2}\Delta(t) - \frac{\zeta'}{\zeta}(0) - \frac{1}{2}\log(1 - e^{-2t})$$

For $0 < t < \log 2$ this becomes

$$\Delta(t) = e^{t/2} - \left(\log 2\pi + \frac{1}{2}\log(1 - e^{-2t})\right)e^{-t/2} > 1 - \log 2\pi - \frac{1}{2}\log 2t$$

Together with Lemma 7 we obtain for $-\log 2 < t < 0$

$$\Delta(t) < \frac{1}{2}\log(-t) + 1.25$$

Especially we have $\Delta(t) < -1$ for $-e^{-4.6} < t < 0$. Now let $T > e^4$ be a real number to be determined later, $M = N(T)$ the number of zeros of ζ with $0 < \Im\rho \leq T$ and $\epsilon = \frac{1}{4\sqrt{M}}$. By Lemma 5 there exists a sequence of real numbers s_i, $1 \leq i \leq N$ satisfying $s_1 \geq 1$, $s_{i+1} \geq s_i + 1$ and

$$s_N \leq \frac{N(32\pi^2 M)^{M/2}\Gamma(M/2)}{\pi^{M/2}} < Ne^{\frac{3}{2}M\log M + 4M}, \tag{1}$$

such that

$$\left(\sum_\rho{}^* |\arg s\gamma|\right)^2 \leq M\sum_\rho{}^* |\arg s\gamma|^2 \leq \frac{1}{2\sqrt{2}}$$

where $\arg z$ is chosen to lie in the interval $[-\pi, \pi]$. Note that with this choice we have $|\arg s\gamma| \leq 2\pi\|s\gamma\|$ For each such s_i and every real t we get

$$\begin{aligned}
|\Delta_T(t) - \Delta_T(t+s_i)|^2 &\leq \left(\sum_{|\gamma|\leq T}\left|\frac{e^{it\gamma}}{\rho} - \frac{e^{i(t+s_i)\gamma}}{\rho}\right|\right)^2 \\
&\leq \left(\sum_{0\leq\gamma\leq T}\left|\frac{\arg s\gamma}{\rho}\right|\right)^2 \\
&\leq \frac{1}{\gamma_0}\left(2\sum_{0\leq\gamma\leq T}|\arg s\gamma|\right)^2 \\
&\leq \frac{1}{2\gamma_0} \\
&= \frac{1}{28.269\ldots}
\end{aligned}$$

Now assume that $\Delta(t+s_i) > -1.01$ for all t with $-e^{-4.6} < t < 0$. Then on one hand we get

$$\begin{aligned}
\int_{e^{-4.6}}^{0} |\Delta(t+s_i) - \Delta(t)|^2 dt &< \int_{e^{-4.6}}^{0} |\Delta(t) - \Delta_T(t)|^2 dt + \int_{e^{-4.6}}^{0} |\Delta_T(t+s_i) - \Delta_T(t)|^2 dt \\
&\quad + \int_{e^{-4.6}}^{0} |\Delta_T(t+s_i) - \Delta(t+s_i)|^2 dt \\
&< \frac{4}{9}\frac{\log^3 T}{T} + \frac{e^{-4.6}}{196}
\end{aligned}$$

while on the other hand we have

$$\begin{aligned}\int_{e^{-4.6}}^{0} |\Delta(t+s_i)-\Delta(t)|^2 dt &> -\int_{0}^{e^{-4.6}} (0.5\log t - 2.26)^2 dt \\ &> 0.32e^{-4.6}\end{aligned}$$

These estimates contradict each other, provided that $\frac{4}{9}\frac{\log^3 T}{T} < 0.31e^{-4.6}$, i.e. for $T > 282000$. Thus we get $M < 590000$, and from (1) we conclude that $s_N < N \cdot e_2(16.6)$. Now if $t > 10$ then $\Delta(t) < -1.01$ implies $\Psi(e^t) > e^t + 1.01e^{t/2} - 2$ and by Lemma 6 the latter implies $\pi(e^t) > li\ e^t$, provided that $t > 60$. Since there are at most 60 values s_i excluded by the last condition, we see that in the interval $[2, exp(N \cdot e_2(16.2))]$ there are at least $N - 60$ values x_i, such that $x_{i+1} > e \cdot x_i$, and $\pi(x_i) > li\ x_i$. Since

$$\int_a^{e\cdot a} \frac{\Psi(e^t)-e^t}{e^{t/2}} dt < \sum_\rho \frac{2}{|\gamma\rho|} < 0.1$$

between x_i and x_{i+1} there is some y_i such that $\pi(y_i) < li\ y_i$. Hence in the interval $[2, exp(N \cdot e_2(16.2))]$ there are at least $N - 60$ sign changes of $\pi(x) - li\ x$. Our claim now follows from the fact that $61 \cdot e_2(16.6) < e_2(16.7)$.

4 Lemmata for Theorem 1

Fix a natural number $q > 2$, and assume that no L-series $\pmod q$ vanishes in $\Re\, s > \frac{1}{2}$. In the sequel let χ be any charakter $\pmod q$. We will prove Theorem 1 under the additional assumption that $q > e(1260)$, it will be apparent from the proofs that stronger conclusions than Theorem 1 can be obtained in the case of small values of q, however, we do not believe that these results are worth the additional effort.

Define the functions $\Delta(t,\chi) = \sum_\gamma \frac{e^{it\gamma}}{\rho}$ and $\Delta_T(t,\chi) = \sum_{|\gamma|<T} \frac{e^{it\gamma}}{\rho}$, where both summations run over the nontrivial roots of $L(s,\chi)$.

Lemma 8. *Denote with $N(T,\chi)$ the number of zeros of $L(s,\chi)$ with $0 < \Re\, \rho < 1, |\Im\, \rho| < T$. Then for $q, T > 10$ we have*

$$\left| N(T,\chi) - \frac{T}{\pi}\log\frac{qT}{2\pi} + \frac{T}{\pi} \right| < \frac{1}{2.1}\log qT + 30$$

For $\log qT > 1260$ and $q, T > 40$ the bounds

$$N(T,\chi) < \frac{1}{3} T \log qT$$

and

$$N(T+1,\chi) - N(T,\chi) < \log qT.$$

Let $N_+(T,\chi)$ denote the number of zeros with $0 \leq \gamma \leq T$, and $N_-(T,\chi)$ the number of zeros with $0 \geq \gamma \geq -T$. Then we have for $q, T > 40$ and $\log qT > 1260$ the bound

$$|N_+(T,\chi) - N_-(T,\chi)| < \frac{5}{4}\log qt.$$

Finally, we have

$$\sum_\rho \frac{1}{|\rho|^2} \leq 13\log q. \tag{2}$$

Proof. The asymptotic bound for $N(T,\chi)$ follows from [9, Theorem 2.1] setting $\eta = 0.01$. The upper bound for $N(T,\chi)$ follows immediatelly from this estimate. For the upper bound for $N(T+1,\chi) - N(T,\chi)$ we begin with the equation

$$-\Re\frac{L'}{L}(s,\chi) = \frac{1}{2}\log\frac{q}{\pi} + \frac{1}{2}\Re\frac{\Gamma'}{\Gamma}\left(\frac{s+a}{2}\right) - \Re\sum_\rho \frac{1}{s-\rho}, \tag{3}$$

where the summation over the zeros has to be taken with respect to increasing imaginary part, and $a = \frac{1-\chi(-1)}{2}$. To bound the term coming from the Γ-function, we use the estimate (see[1, 6.1.42])

$$\left|\log\Gamma(z) - \left(z - \frac{1}{2}\right)\log z + z - \frac{1}{2}\log 2\pi - \frac{1}{12z}\right| \leq \frac{K(z)}{360|z^3|},$$

where $K(z) = \sup_{u\in\mathbf{R}}\left|\frac{z^2}{z^2+u^2}\right|$, which for $\Re z \in [5/4, 7/4]$ and $\Im z > 40$ implies

$$\left|\frac{\Gamma'}{\Gamma}(z) - \log z\right| \leq \frac{1}{79},$$

which together with (3) implies

$$-\Re\frac{L'}{L}(5/4+it,\chi) \leq \frac{1}{2}\log qt - \Re\sum_\rho \frac{1}{s-\rho} - \frac{1}{2}.$$

Set $t = T+1/2$, and assume that $N(T+1,\chi) - N(T,\chi) > \frac{5}{4}\log qT$. Then every zero with imaginary part in the range $[T, T+1]$ would contribute at least $\frac{12}{13}$ to the right-hand side sum, and the last inequality would imply

$$-\Re\frac{L'}{L}(5/4+it,\chi) \leq -\frac{2}{13}\log qT \leq -193,$$

which would contradict the lower bound

$$-\Re\frac{L'}{L}(5/4+it,\chi) \geq \frac{\zeta'}{\zeta}(5/4) \geq$$

Finally, the bound comparing $N_+(T,\chi)$ and $N_-(T,\chi)$ can be proven in the same way as [9, Theorem 2.1], and the bound for $\sum_\rho \frac{1}{|\rho|^2}$ follows from the other estimates. □

Just as in section 2 we get

Lemma 9. *Let $a > b > 0$ be real numbers with $a - b < \frac{1}{324}$ and $T > e^4$. Set $\Delta_T(t) = \sum_{|\gamma|>T} \frac{e^{it\gamma}}{1/2+i\gamma}$. Then we have*

$$\int_a^b |\Delta_T(t)|^2 dt = \sum_{|\gamma_1|,|\gamma_2|>T} \frac{1}{(1/2+i\gamma_1)(1/2+i\gamma_2)} \frac{e^{b(\gamma_1+\gamma_2)} - e^{a(\gamma_1+\gamma_2)}}{\gamma_1+\gamma_2} < \frac{2}{9}\frac{\log^3 qT}{T}$$

For $x > 1$ we have the explicit formula

$$\Psi(x,\chi) = E_\chi x - \sqrt{x}\sum_\rho \frac{e^{i\gamma \log x}}{\rho} - d_\chi \log x - R(x,\chi) + B(\chi)$$

where

$$\begin{aligned} E_\chi &= \begin{cases} 1 & \text{if } \chi = \chi_0 \\ 0 & \text{otherwise} \end{cases} \\ d_\chi &= \begin{cases} 1 & \text{if} \quad \chi(-1) = 1, \chi \neq \chi_0 \\ 0 & \text{if} \quad \chi(-1) = -1 \text{ or } \chi = \chi_0 \end{cases} \\ R(x,\chi) &= \begin{cases} \frac{1}{2}\log(1-x^{-2}) & \text{if} \quad \chi(-1) = 1 \\ \frac{1}{2}\log(1-x^{-2}) + \log\frac{x}{x+1} & \text{if} \quad \chi(-1) = -1 \end{cases} \\ B(\chi) &= -E_\chi + \log 2 - C + \log\frac{q}{\pi} + \frac{L'}{L}(1,\bar{\chi}) \end{aligned}$$

The value of $B(\chi)$ can be obtained using the functional equation, see [6, Lemma 1]. Define $\Delta(t,q,a) := \frac{1}{\varphi(q)}\sum_\chi \chi(a)\Delta(t,\chi)$. To estimate $\Delta(t,q,a)$ in a neighbourhood of 0, we need an upper bound for $B(\chi)$, and hence for $\frac{L'}{L}(1,\chi)$.

Lemma 10. *Let $q > 10$ be an integer, and χ a character $\pmod q$. Then there is some constant θ of absolute value at most 1, such that*

$$\left|\sum_\chi \overline{\chi(a)}\frac{L'}{L}(1,\chi)\right| = \frac{\varphi(q)\Lambda(a)}{a} + \vartheta\big(2\log^2 q + 9\sqrt{\varphi(q)\log q}\big).$$

Proof. The proof will be similar to the estimate given by Masley and Montgomery[8], however, things become easier since we assume GRH here. Set $f(s) = \sum_\chi \overline{\chi(a)}\frac{L'}{L}(s,\chi)$, thus we have to estimate $f(1)$. Using the Brun-Titchmarsh inequality we get for $\sigma > 1$ the estimate

$$\left|f(\sigma) - \frac{\Lambda(a)\varphi(q)}{a^\sigma}\right| < \frac{\Lambda(q+a)\varphi(q)}{(q+a)^\sigma} + 3 + \frac{1}{(\sigma-1)\log 2} + \log^2 q$$

(see [8], Lemma 1). Now differentiating the partial fraction decomposition of $\frac{L'}{L}$ we get for $\sigma > 1$

$$f'(\sigma) = \sum_\chi \overline{\chi(a)} \sum_\rho \frac{1}{(\sigma-\rho)^2} + \vartheta$$

where the inner sum runs over all nontrivial zeros of $L(s,\chi)$ and $|\vartheta| < 1$. We assume $\Re\,\rho = \frac{1}{2}$ for all ρ, so the inner sum can be estimated using lemma 3 by $10\log q$, thus $|f'(\sigma)| < 10\varphi(q)\log q + 1$. Finally

$$\left|f(1) - \frac{\Lambda(a)\varphi(q)}{a^\sigma}\right| < 3+\log^2 q+\log q+\frac{1}{(\sigma-1)\log 2}+10(\sigma-1)\varphi(q)\log q+(\sigma-1)$$

Choosing $\sigma = 1 + \frac{1}{\sqrt{7\varphi(q)\log q}}$ we obtain

$$\left|f(1) - \frac{\Lambda(a)\varphi(q)}{a}\right| < 2\log^2 q + 8\sqrt{\varphi(q)\log q},$$

which proves our claim. □

Now we have enough information to give an estimate for $\Delta(t,q,a)$ for t close to 0.

Lemma 11. *For $0 < t < \log 2$, $q > e^{32}$ we have for some real θ satisfying $|\theta| < 1$ the estimate*

$$\Delta(t,q,1) = \left(\log q - \frac{1}{2}\log(1-e^{-2t}) + 2\theta\right)e^{-t/2},$$

and for $a \not\equiv 1 \pmod q$ we have the bound

$$|\Delta(t,q,a)| \leq 3$$

Proof. We consider three cases: $a \equiv 1 \pmod q$, $a \equiv -1 \pmod q$ and $a \not\equiv \pm 1 \pmod q$.

For $a \not\equiv 1 \pmod q$, all contributions to $\Delta(t,\chi)$, which are independent of χ cancel, if further $a \not\equiv -1 \pmod q$ terms depending only on $\chi(-1)$ cancel as well, so if $a \not\equiv \pm 1 \pmod q$ we get for $0 < t < \log 2$

$$\begin{aligned}\Delta(t,q,1) &= \frac{e^{t/2}-e^{-t/2}}{\varphi(q)} + \frac{te^{-t/2}}{\varphi(q)} + \frac{e^{-t/2}}{\varphi(q)}\sum_\chi \overline{\chi(a)}\frac{L'}{L}(1,\bar\chi)\\ &= \frac{e^{t/2}-e^{-t/2}}{\varphi(q)} + \frac{te^{-t/2}}{\varphi(q)} + \frac{\Lambda(a)e^{-t/2}}{a} + \frac{\vartheta e^{-t/2}}{\varphi(q)}\left(2\log^2 q + 8\sqrt{\varphi(q)\log q}\right).\end{aligned}$$

For $a \equiv -1 \pmod q$ we get

$$\begin{aligned}\Delta(t,q,-1) &= \frac{e^{t/2}}{\varphi(q)} + \frac{(\varphi(q)/2-1)te^{-t/2}}{\varphi(q)} + \frac{1}{2}\log\frac{e^t}{e^t+1} + \frac{e^{-t/2}}{\varphi(q)}\sum_\chi \overline{\chi(a)}\frac{L'}{L}(1,\bar\chi)\\ &= \frac{e^{t/2}}{\varphi(q)} + \frac{(\varphi(q)/2-1)te^{-t/2}}{\varphi(q)} + \frac{1}{2}\log\frac{e^t}{e^t+1} + \frac{\vartheta e^{-t/2}}{\varphi(q)}\left(2\log^2 q + 8\sqrt{\varphi(q)\log q}\right).\end{aligned}$$

Finally for $a \equiv 1 \pmod q$ we get

$$\begin{aligned}
\Delta(t,q,1) &= \frac{e^{t/2}-e^{-t/2}}{\varphi(q)} + \frac{(\varphi(q)/2-1)te^{-t/2}}{\varphi(q)} + \frac{e^{-t/2}}{\varphi(q)}\sum_\chi \overline{\chi(a)}\frac{L'}{L}(1,\bar\chi) \\
&\quad + e^{-t/2}\big(\log 2 - C + \log\frac{q}{\pi} - \frac{1}{2}\log(1-e^{-2t})\big) \\
&= \frac{e^{t/2}-e^{-t/2}}{\varphi(q)} + \frac{(\varphi(q)/2-1)te^{-t/2}}{\varphi(q)} + e^{-t/2}\big(\log 2 - C + \log\frac{q}{\pi} - \frac{1}{2}\log(1-e^{-2t})\big) \\
&\quad + \frac{\vartheta e^{-t/2}}{\varphi(q)}\left(2\log^2 q + 8\sqrt{\varphi(q)\log q}\right).
\end{aligned}$$

For $q > 6$ we have $\varphi(q) > \sqrt{q}$, using this together with the bound $q > e^{32}$ we can conclude that the terms involving θ are of absolute value ≤ 0.02, and all the other terms with the exception of $\frac{1}{2}\log(1-e^{-2t})$ and $\log q$ can easily be bounded absolutely. Putting these bounds together, we obtain our claim. □

Lemma 12. *We have for* $|x| \leq 0.01$ *and* $q \geq \exp(1260)$ *the bounds*

$$\left|\int_0^x \Delta(t,\chi) + \Delta(-t,\chi)\, dt\right| < 53x\log q$$

and

$$\left|\int_0^x \Delta(t,q,a) + \Delta(-t,q,a)\, dt\right| < 53x\log q.$$

Proof. It suffices to prove the first inequality, since the second is obtained by averaging over all characters. Denote with ρ_n the n-th zero of $L(s,\chi)$ with positive imaginary part, ρ_{-n} the n-th zero with negative imaginary part. By Lemma 8 we have $|\gamma_n - \gamma_{-n}| < 1$. Further we have

$$|\Delta(t,\chi) + \Delta(-t,\chi)| = \left|\sum_\rho \frac{e^{t\gamma_n} + e^{-t\gamma_n} + e^{t\gamma_{-n}} + e^{-t\gamma_{-n}}}{\rho}\right|,$$

and each single summand can be estimated as follows.

$$\begin{aligned}
\frac{e^{it\gamma_n}+e^{-it\gamma_n}}{\rho_n} + \frac{e^{it\gamma_{-n}}+e^{-it\gamma_{-n}}}{\rho_{-n}} &= \frac{e^{it\gamma_n}+e^{-it\gamma_n}}{|\rho|^2} - (e^{it\gamma_n}+e^{-it\gamma_n})\left(\frac{1}{\overline{\rho_n}} - \frac{1}{\rho_{-n}}\right) \\
&\quad + \frac{1}{\rho_{-n}}\big((e^{it\gamma_{-n}}+e^{-it\gamma_{-n}}) - (e^{it\gamma_n}+e^{-it\gamma_n})\big) \\
&\leq \frac{4}{|\rho_n|^2} + \frac{1}{\rho_n}\min(4,2t),
\end{aligned}$$

since

$$\begin{aligned}
|e^{-it\gamma_{-n}}) - e^{-it\gamma_n})| &= |e^{-it\gamma_{-n}-it\gamma_n} - 1| \\
&< \min(2, t\gamma_{-n} + t\gamma_n).
\end{aligned}$$

We will use this estimate for small values of γ_n. For large values of γ_n we estimate the integral of a single term by

$$\left|\int_0^x e^{it\gamma_n}\,dt\right| \leq \frac{2}{|\gamma_n|}.$$

Putting these two estimates together and using (2), we obtain

$$\begin{aligned}
\left|\int_0^x \Delta(t,\chi)+\Delta(-t,\chi)\,dt\right| &\leq \sum_n \min\left(\frac{4x}{|\rho_n|^2}+\frac{x^2}{|\rho_n|}, \frac{4}{\gamma_n\rho_n|}\right)\\
&\leq \sum_n \frac{4x}{|\rho_n|^2} + \sum_{\gamma_n<x^{-2}} \frac{x^2}{|\rho_n|} + \sum_{\gamma_n\geq x^{-2}} \frac{4}{\gamma_n|\rho_n|}\\
&\leq 52x\log q + \sum_{1\leq n\leq x^{-2}} \frac{5x^2\log\big(q(n+1)\big)}{3n} + \sum_{n\geq x^{-2}} \frac{5\log\big(q(n+1)\big)}{3n^2}\\
&\leq 52x\log q + 2x^2(2\log(x^{-1})+1)(\log q+2\log(x^{-1})+1)\\
&\quad +2x^2\log q + 2x^2\log(x^{-1})\\
&\leq 53x\log q,
\end{aligned}$$

provided that $x<0.01$ and $\log q>100$, hence our claim. □

The next lemma allows us to translate a statement on $\Psi(x,q,1)-\Psi(x,q,a)$ into a statement on $\pi(x,q,1)-\pi(x,q,a)$.

Lemma 13. *Let $q>\exp(1260)$ be an integer, $x>\exp(27q\log q)$ be a real number such that $\Psi(x,q,1)-\Psi(x,q,a)>\frac{7f(q)}{\varphi(q)}\sqrt{x}$, where $f(q)$ is the number of solutions of the congruence $x^2\equiv 1 \pmod q$. Then we have $\pi(x,q,1)>\pi(x,q,a)$. On the other hand, is a is a quadratic nonresidue, and $\Psi(x,q,1)<\Psi(x,q,a)+\frac{\sqrt{x}}{\varphi(q)}$, we have $\pi(x,q,1)<\pi(x,q,a)$.*

Proof. As in the proof of Lemma 6, we have for $x>e^{12}$ the relation

$$\Pi(x,q,1)-\Pi(x,q,a) = \frac{\Psi(x,q,1)-\Psi(x,q,a)}{\log x} + \frac{2\theta\sqrt{x}}{\varphi(q)\log x}\left(\frac{1}{\log x}\sum_\rho \frac{1}{|\rho|^2} + x^{-1/6}\log x\right)$$

with some θ satisfying $|\theta|<1$. Using (2) we obtain

$$\Pi(x,q,1)-\Pi(x,q,a) \geq \frac{\Psi(x,q,1)-\Psi(x,q,a)}{\log x} - \frac{27\log q}{\log^2 x}\sqrt{x}.$$

On the other hand, using the Brun-Titchmarsh inequality to estimate the contribution of higher powers to $\Pi(x,q,1)$, we obtain for $x>q^8$ the estimate

$$\Pi(x,q,1)-\Pi(x,q,a) \leq \pi(x,q,1)-\pi(x,q,a) + \frac{6f(q)\sqrt{x}}{\varphi(q)\log q} + x^{1/3}.$$

Putting these estimates together, we get for $q>\exp(1260)$ and $x>\exp(27q\log q)$ the first estimate of our lemma. The proof of the second estimate is similar, yet somewhat easier. □

5 Proof of Theorem 1

The proof begins as the proof of Theorem 2. By Lemma 11, we have for $0 < t < \log 2$

$$\left|\Delta(t, q, 1) - e^{-t/2}(\log q - \frac{1}{2}\log(1 - e^{-2t}))\right| < 2,$$

as well as

$$|\Delta(t, q, a)| < 3$$

for $-1 < t < 1$, $(a, q) = 1$, and $a \not\equiv 1 \pmod q$. Applying Lemma 12, we obtain for $0 < x \leq 0.01$ the bound

$$\left|\int_{-x}^{0} \Delta(t, q, 1)\, dt - \int_{0}^{x} e^{-t/2}\left(\frac{1}{2}\log(1 - e^{-2t}) - \log q\right)\, dt\right| < 53x\log q.$$

Setting $x = q^{-120}e^{-15f(q)}$, we deduce that

$$\int_{-x}^{0} \Delta(t, q, 1)\, dt < \int_{-x}^{0} \min_{a\neq 1} \Delta(t, q, 1)\, dt - 4x\log q - 7xf(q). \tag{4}$$

From Lemma 9 we obtain that

$$\int_{-x}^{0} \Delta_T(t, q, 1)\, dt < \int_{-x}^{0} \min_{a\neq 1} \Delta(t, q, 1)\, dt - 3x\log q - 7xf(q),$$

provided that

$$\frac{2\log^3 qT}{9T} < x\log q.$$

The latter condition is satisfied for $T = q^{130}e^{16f(q)}$, provided that $q > e^8$,since $f(q) < q$ holds trivially. From Lemma 8, the number M of zeros occuring in the sum for $\Delta_T(t)$ is at most $qT\log qT \leq q^{140}e^{17f(q)}$. From Lemma 5, applied with $\varepsilon = \frac{1}{4\pi^2 M}$ we obtain a sequence of real numbers $s_1, \dots, s_N$, such that $s_1 \geq 1$, $s_{i+1} \geq s_i + q^3$,

$$\begin{aligned} s_N \leq \frac{q^3 N(8\pi^2 M)^M}{\pi^{M/2}} &< \exp\left(\frac{3}{2}M\log M + 3M\right) \\ &< \exp\left(q^{150}e^{18f(q)}\right) \end{aligned}$$

and

$$\left(\sum_{\rho}{}^{*} |\arg s_i\gamma|\right)^2 \leq M\sum_{\rho}{}^{*} |\arg s_i\gamma|^2 \leq 1,$$

where summation runs over all nontrivial zeros of all L-series $\pmod q$ with imaginary part γ satisfying $|\gamma| \leq q^{130}e^{16f(q)}$. As in Section 3, this bound implies

$$|\Delta_T(t, q, a) - \Delta_T(t + s_i, q, a)| \leq 2 < \log q \tag{5}$$

for all $(q,a)=1$ and $i=1,\dots,N$. Now assume that for all $t\in[-x,0]$ we had

$$\Delta(t+s_i,q,1) > \min_{a\neq 1}\Delta(t+s_i,q,a)-\log q-7f(q). \tag{6}$$

Then we get on one hand from (4) and Lemma 9 the estimate

$$\int_{-x}^{0}|\Delta(t+s_i,q,1)-\Delta(t,q,1)|\,dt < 2x\sqrt{\log q}+2x,$$

whereas on the other hand we have from (3) and Lemma 9, applied to $\Delta(t,q,a)$ the bound

$$\int_{-x}^{0}|\Delta(t+s_i,q,1)-\Delta(t,q,1)|\,dt > 3x\log q-2x\sqrt{\log q}-2x,$$

yielding a contradiction for $q>e^2$. Hence, for each i, there is some $t\in[-x,0]$, such that (5) fails for this value of t, and from Lemma 13 we deduce that this implies

$$\pi(e^{t+s_i},q,1)\geq\pi(e^{t+s_i},q,a)$$

for all $a\not\equiv 1 \pmod q$, provided that $s_i>27q\log q$.

Repeating the same argument, this time starting with the inequality

$$\int_{-x}^{0}\Delta(t,q,1)\,dt < \int_{-x}^{0}\min_{a\neq 1}\Delta(t,q,1)\,dt-4x\log q-7xf(q)$$

instead of (3), we find that for each s_i there is some $t\in[s_i,s_i+x]$ such that

$$\pi(e^{t+s_i},q,1)\leq\pi(e^{t+s_i},q,a)$$

for all $a\not\equiv 1 \pmod q$. Hence, there are at least $N-27q\log q$ sign changes of $\pi(x,q,1)-\max_{a\neq 1}\pi(x,q,a)$ below $\exp(N\exp(q^{150}e^{18f(q)}))$, solving for N yields the second statement of Theorem 1, since

$$27q\log q\exp(q^{150}e^{18f(q)})\leq\exp(q^{160}e^{18f(q)}).$$

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