THE ASYMPTOTIC BEHAVIOUR OF THE NUMBER OF TREES IN CERTAIN CLASSES

JAN-CHRISTOPH SCHLAGE-PUCHTA

Abstract. When counting combinatorial structures, asymptotic questions are often encoded into the singularity of least absolute value of the corresponding generating function. In this article we describe the rate of convergence of this singularity, as the structures run over a sequence of sets of trees, which in the limit exhaust the set of all trees. This question is motivated by applications in proof theory, more precisely, phase transitions in unprovability results.

1. 1. Introduction and results

Consider an interesting class $\mathcal{C}$ of finite rooted trees. A natural question to ask is how many trees with a given number of nodes are contained in this class. Of course the answer depends not only on the class $\mathcal{C}$, but also on the question which trees are to be regarded as equal; for example we can count trees up to isomorphism, up to cyclic permutation of the daughter trees of each node, or we can completely fix the ordering of the daughter trees. Further, if we consider r-ary trees we may or may not allow for empty daughter trees.

In view of these possibilities it appears reasonable to restrict oneself to problems stemming from real applications. However, it turns out that in many important cases one can give a nice description of the generating series $T_\mathcal{C}(x)$ of the number of trees in $\mathcal{C}$, which can be used to determine the asymptotic behaviour of the number of trees by complex analytic means. If $T_\mathcal{C}(x)$ satisfies an algebraic equation, then the location and the order of the singularity of this equation of least absolute value determines the asymptotic behaviour with great precision. Hence, the determination of this singularity is one of the first things to do.

In this article we consider the question as to how the singularity behaves as $\mathcal{C}$ varies.

Our first result deals with the number of r-ary trees for varying r. We denote by the degree of a node the number of daughter nodes, that is, unless the node is the root, one less than the graph theoretic degree.

Theorem 1.- *Let $t_k(n)$ be the number of trees with n nodes and degree at most k, $t(n)$ be the of all trees with n nodes. Put $T_k(x) = \sum_{n\geq 1} t_k(n)x^n$, and $T(x) = \sum_{n\geq 1} t(n)x^n$. Then T_k has a real branch point of order 2 at ρ_k, and no other singularity of absolute value $\leq \rho_k + \delta$ for some positive constant δ. We have $\rho + e^{-c_1 k} < \rho_k < \rho + e^{-c_2 k}$ for some constants $c_1 > c_2 > 0$, where ρ is the radius of convergence of $T(x)$.*

One can get the values $c_1 = 2.17$, $c_2 = 0.893$ by just keeping track of the constants in our proof, however, we refrain from giving the details, since doing so involves quite some work, no new ideas, and the numerical value of c_1, c_2 is of dubious interest.

The interesting part here is the dependence of ρ_k on k, the other statements are well known (confer [2], Section IV.7), and are restated here for the convenience of the reader as well as for the definition of ρ_k.

Next we fix r, and consider the set of r-ary planar trees, where each node either has precisely r daughter trees, or is a leaf. Let $s(n)$ be the number of all such trees of height n, and $s_h(n)$ be the number of such trees, such that in each path in the tree the left-most branch is taken at most h times. In particular, $s_0(n)$ is the number of $(r-1)$-ary trees, and $s_1(2) = r$. Then we have the following.

Theorem 2.- *Define $s_h(n)$ as before, and set $S_h(x) = \sum_{n\geq 0} s_h(n)x^n$, and $S(x) = \sum_{n\geq 0} s(n)x^n$. Then S and S_h have a real branch point of order 2 at ρ and ρ_h, respectively, and no other singularity of absolute value $\leq \rho_h + \delta$ for some positive constant δ. Then we have $\rho_h - \rho \asymp h^{-2}$.*

Here $f \asymp g$ means that $f = \mathcal{O}(g)$ and $g = \mathcal{O}(f)$; similarly we shall later use $f \ll g$ to denote $f = \mathcal{O}(g)$. For large h we have the explicit bounds

$$\frac{1}{5(C_1+1)(C_2+1)h^2} < \rho_h - \rho < \frac{56}{C_1 C_2 h^2}$$

where

$$C_1 = \left(\frac{r}{r-1}\right)^{r-2}\binom{r}{2} \qquad C_2 = \left(\frac{r}{r-1}\right)^{r}.$$

Neither the upper nor the lower bound is close to optimal, however, we do not really care about numerical values.

These questions are motivated by problems in proof theory. Bovykin and Weiermann [1] had given analytic statements which are unprovable in Peano arithmetic. More precisely, they gave a parametric version of Kruskal's theorem and showed that as a certain parameter varies there is a sharp phase transition from provable to unprovable (within the frame work of Peano arithmetic), provided that a weak version of Theorem 1 holds true. Weiermann (personal communication) showed that in the situation of Theorem 1 and 2 we have that $\rho_k \to \rho$ holds true, and noted that if one had explicit bounds for the speed of this convergence, one could sharpen these results. In this context Woods [5] showed that the speed of convergence in Theorem 2 is subexponential, however, his proof was indirect and did not give an explicit subexponential bound.

Questions concerning the enumeration of certain types of trees arise repeatedly in the study of phase transitions as introduced by Weiermann [3]. If one works with proof systems different from PA, the question of enumerating trees is replaced by the question of determining for a fixed ordinal α the number of ordinals $\beta < \alpha$, which satisfy $N(\beta) \leq n$, where N is a suitable complexity measure, e.g. the length of the Cantor normal form of β. In general the generating series of these counting functions do not satisfy any reasonable functional equation, which makes the related problems much harder. However, the case $\alpha = \omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega}}}}}$ was successfully treated by Weiermann [4].

One advantage of reducing problems to the enumeration of trees is that this area of analytic combinatorics is well developed. The specific question on the convergence speed of the singularities ρ_h might look artificial to a combinatorist, still we can use the machinery described in [2], in particular chapter II and VI.

2. 2. Proof of Theorem 1

It is well known that the series $T(x)$ is the unique solution of the functional equation

$$T(x) = x \exp\left(\sum_{\nu\geq 1} \frac{T(x^\nu)}{\nu}\right)$$

satisfying $T(0) = 0$ and $T'(0) = 1$. Our first result shows that T_k can be defined as a solution of an approximation of this equation. Define the polynomial $P_k(X_1, \dots, X_k)$ as follows. Expand $\exp\left(\sum_{\nu\geq 1} \frac{X_\nu}{\nu}\right)$ into a power series in the infinitely many variables $X_1, \dots$, and let P_k be the part consisting only of monomials $X_1^{a_1} \dots X_m^{a_m}$ satisfying $\sum j a_j \leq k$. The following is essentially [2, Prop. I.6], note that while in the statement of that proposition only the existence of some polynomial is shown, the precise form is contained in the proof.

Lemma 2.1.- *The function T_k is the unique solution of the functional equation $T_k(x) = xP_k(T_k(x), T_k(x^2), \dots, T_k(x^k))$.*

Lemma 2.2.- *The least positive singularity of T_k is given by the system of equations*

$$\begin{aligned} T_k(\rho_k) &= \rho_k P_k\big(T_k(\rho_k), \dots, T_k(\rho_k^k)\big) \\ 1 &= \rho_k \frac{\partial}{\partial x_1} P_k\big(T_k(\rho_k), \dots, T_k(\rho_k^k)\big). \end{aligned}$$

Proof.- This follows from the implicit function theorem, see [2, Chapter VII.4]. QED

Since $P_k(x_1, \dots, x_k)$ consists of the monomials of weight $\leq k$ in $\exp(\sum_i \frac{x_i}{i})$, and the partial derivative of $\exp(x_1 + f(x_2, \dots, x_k))$ with respect to x_1 is again $\exp(x_1 + f(x_2, \dots, x_k))$, we see that

$$\frac{\partial}{\partial x_1} P_k(x_1, \dots, x_k) = P_{k-1}(x_1, \dots, x_k).$$

Hence, we obtain that

$$T_k(\rho_k) = 1 + \rho_k\Big(P_k\big(T_k(\rho_k), \dots, T_k(\rho_k^k)\big) - P_{k-1}\big(T_k(\rho_k), \dots, T_k(\rho_k^k)\big)\Big). \quad (1)$$

We first prove the upper bound. We set $Q_k = P_k - P_{k-1}$. We have $\rho_k \leq \rho_2 = 0.4202$, $\rho = 0.338$ (Confer [2, p. 455]), in particular, $\rho_k^\ell < \rho$ hold true for all $k, \ell \geq 2$. Clearly, all coefficients in the Taylor series of T_k are at most as large as the corresponding coefficient of T, hence, for $x \in [0, \rho_k]$ we have

$$\begin{aligned} T_k(x) &\leq 1 + \rho_k Q_k\big(T_k(x), T(\rho_2^2) \dots, T(\rho_2^k)\big) \\ &\leq 1 + \rho_k \exp\Big(T_k(x) + \sum_{i=2}^{\infty} \frac{T(\rho_2^i)}{i}\Big) - \rho_k P_{k-1}\big(T_k(x), T(\rho_2^2) \dots, T(\rho_2^k)\big). \end{aligned}$$

We claim that for k sufficiently large this implies $T_k(x) \leq 2$ for all $x \leq \rho_k$. Assume to the contrary that the equation $T_k(x) = 2$ has a solution $x_0 \leq \rho_k$. Then we have

$$2 \leq 1 + \rho_k \exp\Big(2 + \sum_{i=2}^\infty \frac{T(\rho_2^i)}{i}\Big) - \rho_k P_{k-1}\big(2, T(\rho_2^2) \dots, T(\rho_2^k)\big),$$

on the other hand we have

$$\lim_{k\to\infty} P_{k-1}\big(x_1, \dots, x_{k-1}\big) = \exp\Big(\sum_{i=1}^\infty \frac{x_i}{i}\Big)$$

for all non-negative sequences (x_i) for which the series on the right hand side converges, hence, for k sufficiently large we have

$$\left|\exp\Big(2 + \sum_{i=2}^\infty \frac{T(\rho_2^i)}{i}\Big) - P_{k-1}\big(2, T(\rho_2^2) \dots, T(\rho_2^k)\big)\right| < 1,$$

and we obtain $2 < 1 + \rho_k$, contradicting the fact that $\rho_k \leq \rho_2 = 0.4202$. Hence, for k sufficiently large we have $T_k(\rho_k) \leq 2$.

Let $a_1, \dots, a_k$ be non-negative integers satisfying $a_1 + 2a_2 + \dots, +ka_k = k$. The coefficient of $x_1^{a_1} \cdots x_k^{a_k}$ in Q_k is

$$\frac{1}{\big(\sum_{i=1}^k a_i\big)!}\binom{\sum_{i=1}^k a_i}{a_1, \dots, a_k}\prod_{i=1}^k \frac{1}{i^{a_i}} = \frac{1}{\prod_{i=1}^k a_i! i^{a_i}}.$$

Since the coefficients of T are non-negative, we have $T(\rho^\ell) < \rho^{\ell-2}T(\rho^2)$ for all $\rho \geq 2$, since on the other hand $T(\rho^2) > \rho^2$, we obtain $T(\rho^\ell) \leq T(\rho^2)^{\ell/2}$ for all $\ell \geq 2$. The contribution of the monomial corresponding to the tuple $(a_1, \dots, a_k)$ is therefore

$$\frac{2^{a_1}}{a_1!}\prod_{i=2}^k \frac{T(\rho^2)^{\frac{ia_i}{2}}}{a_i! i^{a_i}} \leq \frac{2^{a_1} T(\rho^2)^{\frac{k-a_1}{2}}}{a_1!} = T(\rho^2)^{k/2}\frac{2^{a_1}T(\rho^2)^{\frac{-a_1}{2}}}{a_1!} \ll T(\rho^2)^{\frac{k}{2}}.$$

Monomials in Q_k correspond to partitions of k, hence, the number of such monomials is $< e^{c\sqrt{k}}$, and we obtain that

$$Q_k(T_k(\rho_k), T(\rho^2), \dots, T(\rho^k)) \ll e^{-ck}$$

for some positive c, that is, $T_k(\rho_k) = 1 + \mathcal{O}(e^{-ck})$. From Lemma 2.2 we now deduce

$$\rho_k^{-1} = \big(1 + \mathcal{O}(e^{-ck})\big)P_k(T_k(\rho_k), \dots, T_k(\rho_k^k)). \tag{2}$$

The polynomial P_k consists of a subset of the series $\exp(\sum \frac{x_i}{i})$, hence, in the domain $x_1 \leq 2$, $x_\ell \leq cT(\rho^2)^{\ell/2}$ the partial derivative of P with respect to any of the x_i is bounded independent of k. Hence,

$$P_k(T_k(\rho_k), \dots, T_k(\rho_k^k)) - P_k(1, T(\rho_k^2), \dots, T(\rho_k^k)) \ll |1 - T_k(\rho_k)| + \sum_{i=2}^\infty |T(\rho_k^i) - T_k(\rho_k^i)|$$

The series T and T_k coincide in the first k coefficients, hence, for real positive x the difference $|T(x) - T_k(x)|$ can be bounded by the difference of $T(x)$ and the partial sum consisting of the terms of degree $\leq k$. Hence, $|T(x) - T_k(x)| \ll (x/\rho)^k$ uniformly for $x \leq \rho$, and we obtain

$$\begin{aligned}
P_k(T_k(\rho_k), \dots, T_k(\rho_k^k)) - P_k(1, T(\rho_k^2), \dots, T(\rho_k^k)) \\
&\ll e^{-ck} + \sum_{i=2}^{\infty} \big(\frac{\rho_k^i}{\rho}\big)^k \\
&= e^{-ck} + \frac{\rho_k^{2k}}{\rho^k} \cdot \frac{1}{1-\rho_k} \ll e^{-ck} + \left(\frac{\rho_k^2}{\rho}\right)^k \leq e^{-ck} + \left(\frac{\rho_2^2}{\rho}\right)^k \ll e^{-ck},
\end{aligned}$$

since $\frac{\rho_2^2}{\rho} = 0.5225\ldots < 1$. Note that in order to get the last bound we might have to decrease the value of c. Since ρ_k is bounded against 0 and ∞, we can plug this estimate into (2) to obtain

$$\rho_k^{-1} = \big(1 + \mathcal{O}(e^{-ck})\big) P_k(1, T(\rho_k^2), \dots, T(\rho_k^k)).$$

Since $\rho_k \geq \rho$, replacing ρ_k by ρ decreases the right-hand side, and we obtain

$$\rho_k^{-1} \geq \big(1 + \mathcal{O}(e^{-ck})\big) P_k(1, T(\rho^2), \dots, T(\rho^k)).$$

On the other hand we have

$$\rho^{-1} = \lim_{k\to\infty} P_k(1, T(\rho^2), \dots, T(\rho^k)).$$

Define σ_k as

$$\sigma_k^{-1} = P_k(1, T(\rho^2), \dots, T(\rho^k)).$$

Then $\sigma_k^{-1} - \sigma_{k-1}^{-1} = Q_k(1, T(\rho^2), \dots, T(\rho^k)) \ll e^{-ck}$, hence,

$$|\sigma_k - \rho| = |\sigma_k - \lim_{\ell\to\infty} \sigma_\ell| \ll \sum_{\ell \geq k} e^{-c\ell} \ll e^{-ck},$$

that is, $\rho_k^{-1} \geq \big(1 + \mathcal{O}(e^{-ck})\big)\rho^{-1}$, and we obtain $\rho_k \leq \rho + \mathcal{O}(e^{-ck})$. By choosing c_2 somewhat smaller than c the upper bound follows.

For the lower bound note that (1) implies $T_k(\rho_k) > 1$. On the other hand there exists a tree with $k+2$ vertices which is counted by T, but not by T_k. Hence, $T(x) \geq T_k(x) + x^{k+2}$ holds true for all $x \in [0, \rho]$, in particular $T_k(\rho) \leq 1 - \rho^{k+2}$. This implies $\rho_k \neq \rho$. Differentiating the functional equation yields for $x \in [0, \rho_k)$

$$\begin{aligned}
T_k'(x) &= P_k(T_k(x), \dots, T_k(x^k)) \\
&\quad + x \sum_{j=1}^{k} j x^{j-1} T_k'(x^j) \Big(\frac{\partial}{\partial x_j} P_k\Big)(T_k(x), \dots, T_k(x^k)) \\
&= \frac{T_k(x)}{x} + x P_{k-1}(T_k(x), \dots, T_k(x^{k-1})) \\
&\quad + x \sum_{j=2}^{k} j x^{j-1} T_k'(x^j) \Big(\frac{\partial}{\partial x_j} P_k\Big)(T_k(x), \dots, T_k(x^k)) \\
&\leq \frac{T_k(x)}{2} + x T_k + k T'(\rho_2) \sum_{j=2}^{k} \Big(\frac{\partial}{\partial x_j} P_k\Big)(T_k(x), \dots, T_k(x^k)).
\end{aligned}$$

Since P_k is a polynomial of weight k, differentiating $P_k(x_1, \dots, x_k)$ with respect to x_i increases the value by $\frac{k}{x_i}$ at most, and we obtain

$$\Big(\frac{\partial}{\partial x_j} P_k\Big)(T_k(x), \dots, T_k(x^k)) \leq \frac{k}{T_k(x^j)} P_k((T_k(x), \dots, T_k(x^k))$$

$$\leq \quad \frac{jT_k(x)}{x^{j+1}} \leq \frac{2k}{x^{k+1}},$$

and therefore $\frac{T_k'(x) \ll k^3}{x^{k+1}}$. Since $T(\rho_k) - T(\rho) > \rho^{k+2}$, we deduce

$$\rho_k - \rho \gg \frac{\rho^{k+2}\rho_k^{k+1}}{k^3} \gg e^{-ck},$$

and the claimed lower bound follows as well.

3. 3. Proof of Theorem 2

We begin by recalling some properties of S_h.

Proposition 3.-

(1) *We have* $S_0(z) = 1$, $S_{h+1}(z) = 1 + zS_h(z)S_{h+1}(z)^{r-1}$;
(2) *For* $h \geq 2$ *we have* $S_h(\rho_h) = \frac{r-1}{r-2}$;
(3) *We have* $S(\rho) = \frac{r}{r-1}$;
(4) *We have* $\rho = \frac{(r-1)^{r-1}}{r^r}$.

Proof.- The recursive relation follows from the definition of the h-th family. The second and third claim follows by singularity analysis from this. The last statement was proven by Weiermann. Note that here the determination of the value of ρ is difficult; once the value of ρ is known, it can be verified by a tedious yet trivial calculation. QED

For a real number $x \in [0, \rho_m]$ we put $\delta_m = \frac{r}{r-1} - S_m(x)$. We will study the sequence (δ_m) for fixed x slightly larger than ρ.

Lemma 3.1.- *For every* $\epsilon > 0$ *there exists an* m_0 *and a* $\delta > 0$, *such that* $S_{m_0}(x) \in [\frac{r}{r-1} - \epsilon, \frac{r}{r-1}]$ *for all* $x \in [\rho, \rho + \delta]$.

Proof.- We have $S_m(\rho) \to S(\rho) = \frac{r}{r-1}$, hence we can choose m_0 such that $S_{m_0}(\rho) \geq \frac{r}{r-1} - \epsilon$. Since S_{m_0} is a power series with non-negative coefficients, it is non-decreasing, and we obtain $S_{m_0}(x) \geq \frac{r}{r-1} - \epsilon$ for all $x \in [\rho, \rho_m]$. On the other hand there exist trees which are counted by S, and not by S_{m_0}, hence $S_{m_0}(\rho) < S(\rho)$, and since both S_{m_0} and S are continuous, this inequality holds true in some neighbourhood of ρ. Hence if δ is sufficiently small, then $S_{m_0}(x) < \frac{r}{r-1}$ for $x < \rho+\delta$. QED

Lemma 3.2.-*If* $x \in [\rho, \rho_{m+1}]$ *and* $\delta_m > er(x-\rho)$, *then*

$$\delta_{m+1} = \delta_m - C_1\delta_m^2 - C_2(x-\rho) + \mathcal{O}(\delta_m^3 + \delta_m(x-\rho))$$

with

$$C_1 = \left(\frac{r}{r-1}\right)^{r-2}\left(\binom{r}{2}\right), \qquad C_2 = \left(\frac{r}{r-1}\right)^{r}.$$

Proof.- Inserting the definition of δ_m into the recurrence relation for T_m we obtain

$$\begin{aligned} \frac{r}{r-1} - \delta_{m+1} \quad &= \quad 1 + \rho\left(\frac{r}{r-1} - \delta_m\right)\left(\frac{r}{r-1} - \delta_{m+1}\right)^{r-1} \\ &\quad + (x-\rho)\left(\frac{r}{r-1} - \delta_m\right)\left(\frac{r}{r-1} - \delta_{m+1}\right)^{r-1} \end{aligned}$$

We claim that given $\delta_m \in [e(x-\rho), 1]$ this equation has a unique solution $\delta_{m+1} \in [0, \delta_m]$. View the difference of the right and the left hand side as a function of δ_{m+1}.

If we insert $\delta_{m+1} = 0$, we get

$$\frac{r}{r-1} - 1 - \rho\left(\frac{r}{r-1} - \delta_m\right)\left(\frac{r}{r-1}\right)^{r-1} - (x-\rho)\left(\frac{r}{r-1} - \delta_m\right)\left(\frac{r}{r-1}\right)^{r-1}$$
$$> \frac{\delta_m}{r} - (x-\rho)\left(\frac{r}{r-1} - \delta_m\right)\left(\frac{r}{r-1}\right)^{r-1} \geq \frac{\delta_m}{r} - \frac{ec}{h^2} \geq 0,$$

while for $\delta_{m+1} = \delta_m$ we have

$$\frac{r}{r-1} - \delta_m - 1 - \rho\left(\frac{r}{r-1} - \delta_m\right)^r - (x-\rho)\left(\frac{r}{r-1} - \delta_m\right)^r$$
$$< \underbrace{\frac{r}{r-1} - 1 - \rho\left(\frac{r}{r-1}\right)^r}_{=0} - \delta_m - (x-\rho)\left(\frac{r}{r-1} - \delta_m\right)^r < 0.$$

The derivative of this function with respect to δ_{m+1} is

$$-1 + (r-1)\rho\left(\frac{r}{r-1} - \delta_m\right)\left(\frac{r}{r-1} - \delta_{m+1}\right)^{r-2}$$
$$+ (r-1)(x-\rho)\left(\frac{r}{r-1} - \delta_m\right)\left(\frac{r}{r-1} - \delta_{m+1}\right)^{r-2}$$
$$< -1 + (r-1)\rho\left(\frac{r}{r-1}\right)^{r-1} + e(r-1)(x-\rho)$$
$$= -\frac{1}{r} + e(r-1)(x-\rho) < 0,$$

and we conclude that $\delta_{m+1} \in [0, \delta_m]$ is uniquely determined by (3).

Inserting the value for ρ into (3), expanding the second term on the right, and bounding the last term on the right rather crudely, we obtain

$$\begin{aligned}\delta_{m+1} &= \delta_m - \left(\frac{r}{r-1}\right)^{r-2}\left(\binom{r-1}{2}\right)\delta_{m+1}^2 - \left(\frac{r}{r-1}\right)^{r-2}(r-1)\delta_{m+1}\delta_m \\ &\quad -(x-\rho)\left(\frac{r}{r-1}\right)^r + \mathcal{O}(\delta_m^3 + \delta_m(x-\rho)),\end{aligned}$$

which leads to

$$\delta_{m+1} \geq \delta_m - C_1\delta_m^2 - C_2(x-\rho),$$

provided that $\delta_m \in [er(x-\rho), \epsilon]$, where $\epsilon > 0$ is a constant depending only on r. In particular $\delta_m - \delta_{m+1} = \mathcal{O}(\delta_m^2 + (x-\rho))$, hence for $\delta_m \in [er(x-\rho), \epsilon]$ we can replace δ_{m+1} on the right by δ_m without increasing the error term, and our claim follows.

If $\delta_m > \epsilon$, the stated error term does not tend to 0 as $m \to \infty$, that is, we can replace the error term by $\mathcal{O}(1)$, and the statement follows from the trivial bounds $\delta_m \leq \frac{1}{r-1}, \delta_{m+1} > -\frac{1}{r-2}$. QED

We now obtain the lower bound for ρ_h. In fact, we show that if we fix $x = \rho + \frac{c}{h^2}$ with c sufficiently small, then $\delta_h > 0$, that is, $S_h(t) < \frac{r}{r-1}$ for $t < x$, and therefore $x < \rho_h$. Choose $\epsilon > 0$ in such a way that the constant implied by the error term in Lemma 3.2 is $< \epsilon^{-1}$. Then we obtain for $\delta_m \in [\frac{ec}{h^2}, \epsilon]$ the recursive bound

$$\delta_{m+1} > \delta_m - (C_1+1)\delta_m^2 - \frac{c(C_2+1)}{h^2}.$$

Choose m_0 in such a way that $\delta_{m_0} < \min(\epsilon, \frac{1}{2(C_1+1)})$, which is possible in view of Lemma 3.1. Put $m_1 = \lceil \delta_{m_0}^{-1} \rceil$. We now prove by induction that

$$\delta_m \geq \frac{1}{2(C_1+1)(m-m_0+1)} - \frac{cm(C_2+1)}{h^2}.$$

For $m = m_0$ this inequality is obviously true. If we assume that it holds for m, then

$$\delta_{m+1} > \frac{1}{2(C_1+1)(m-m_0+1)} - \frac{1}{4(C_1+1)(m-m_0+1)^2} - \frac{c(m+1)(C_2+1)}{h^2}$$

since the right hand side is increasing as a function of δ_m on $[0, \frac{1}{2(C_1+1)}]$. Our claim now follows from the relation

$$\frac{1}{x} - \frac{1}{2x^2} \geq \frac{1}{x} - \frac{1}{x(x+1)} = \frac{1}{x+1}$$

valid for all $x > 1$. We conclude

$$\delta_h > \frac{1}{2(C_1+1)(h-m_0+1)} - \frac{c(C_2+1)}{h} > \frac{1}{4(C_1+1)h} > \frac{erc}{h^2},$$

provided that h is sufficiently large and $c < \frac{1}{5(C_1+1)(C_2+1)}$. Hence we obtain $\rho_h > \rho + \frac{1}{5(C_1+1)(C_2+2)h^2}$.

For the upper bound we show that if c is a sufficiently large constant, then for $x_h = \rho + \frac{c}{h^2}$ the sequence δ_m becomes large and negative quite fast, contradicting the bound $\delta_h \geq -\frac{1}{r-2}$. Assume that $x_h < \rho_h$. Then $T_{m+1}(x_h)$ is given the unique solution of the recursive relation within $[1, \infty)$, that is, we can use (3) without worrying about the existence of a solution. In particular (4) holds true for all δ_m with $m \leq h$, since the restriction $\delta_m > e(x-\rho)$ was only used to prove the existence of the solution of the recursion.

First note that the same computation used for the proof of the lower bound gives $\delta_m < \frac{2}{C_1(m-m_0+m_1)}$, provided that m is sufficiently large. Put $m_2 = \lceil h/3 \rceil$. Then for h sufficiently large we conclude $\delta_{m_2} < \frac{7}{C_1 h}$. Next we use (4) in the weaker form

$$\delta_{m+1} < \delta_m - (C_2-1)(x-\rho),$$

which is valid for $|\delta_m| < \epsilon$. Put $m_3 = \lceil 2h/3 \rceil$. Then

$$\delta_{m_3} < \delta_{m_2} - \frac{hC_2}{4}(x-\rho),$$

hence, if $x - \rho > \frac{56}{C_1 C_2 h^2}$, then $\delta_{m_3} < -\frac{7}{C_1 h}$.

If we expand the right hand side of (3) using the binomial theorem, and assume that δ_m is negative, then all terms are positive. Hence deleting terms of total degree ≥ 3 in δ_m and δ_{m+1} gives a value for δ_{m+1} which is too large, that is, for δ_m negative the error term in (4) is negative. Moreover, $\delta_{m+1} < \delta_m$, and we obtain the inequality

$$\delta_{m+1} < \delta_m - C_1\delta_m^2 - C_2(x-\rho) < \delta_m - C_1\delta_m^2$$

valid for all m such that $\delta_m < 0$ and $m < h$ and all x with $\rho < x < \rho_h$. We now define m_i starting with $i = 4$ by $m_i = m_{i-1} + \lceil 1/(C_1\delta_{m_{i-1}}) \rceil$, this sequence is defined as long as the sequence δ_m is defined, that is, for all i such that $m_i \leq h$. Let $i_{\max}$ be the largest index for which this sequence is defined. Then

$$\delta_{m_{i+1}} < \delta_{m_i} - (m_{i+1} - m_i)C_1\delta_{m_i}^2 \leq \delta_{m_i} - \frac{C_1\delta_{m_i}^2}{C_1\delta_{m_i}} = 2\delta_{m_i}.$$

Since $\delta_h > \frac{r}{r-1} - \frac{r-1}{r-2} = \frac{1}{(r-1)(r-2)} < 1$, we obtain $\frac{7\cdot 2^{i_{\max}}}{C_1 h} < 1$, and therefore $i_{\max} \leq \log h$. On the other hand we have

$$m_{i_{\max}} \leq m_3 + \sum_{i=1}^{i_{\max}-1} \frac{1}{C_1\delta_{m_i}} + 1 \leq \frac{2h}{3} + \frac{2}{C_1\delta_{m_3}} + i_{\max} \leq \frac{20h}{21} + i_{\max}.$$

For h sufficiently large this implies $i_{\max} \geq \frac{h}{22}$, and the two bounds for $i_{\max}$ contradict each other for h sufficiently large. Hence we obtain that the two assumptions $x < \rho_h$ and $x - \rho > \frac{56}{C_1C_2h^2}$ contradict each other, and we conclude $\rho_h \leq \rho + \frac{56}{C_1C_2h^2}$ for h sufficiently large. Hence our theorem follows.

4. 4. Conclusion and outlook

We have considered a filtrations $\mathcal{T}_1 \subset \mathcal{T}_2 \subset \dots$ of the set of trees. We determined the speed of convergence of the dominating singularity of the generating functions of these subsets to the dominating singularity for the set of all trees, and similarly for a certain set of planar trees. In the first case the speed of convergence was exponential, while in the second it was only quadratic. An intuitive explanation for this phenomenon is that in the first case we have that $T_i(\rho_i) \to T(\rho)$, while in the second we have $S_i(\rho_i) = \frac{r-1}{r-2}$, which is bounded away from $S(\rho) = \frac{r}{r-1}$. However, we are not yet able to formulate and prove a strict statement underlying this intuition.

The research in this article is motivated by its application in the theory of logical phase transitions, in fact, an unpublished version of Theorem 1 has already been applied by Bovykin and Weiermann[1].

This leads to further research in two quite different directions. The first is motivated by the analytic behaviour of the generating functions. Suppose F is a function in two variables, analytic in a neighbourhood of the origin. Assume there exists a unique sequence of functions (f_i) satisfying $f_0 = 1$, $F(f_{i+1}, f_i) = 0$. Under what conditions on F does there exist a unique function f satisfying $F(f, f) = 0$, and how does the sequence (f_i) approximate f? Similarly if F_i is a series of functions in one variable, approximating a function F in a suitable way, are there reasonable conditions such that the solutions f_i of the functional equations $F_i(f_i) = 0$ approximate the solution f of $F(f) = 0$?

The second direction of research is motivated by the proof theoretic applications. If one wants to generalize the phase transitions obtained in [1] to other systems of arithmetic, one has to solve counting problems involving ordinal numbers below a certain ordinal and with bounded complexity. This line of research appears to be more demanding then the first, as in general we cannot expect that the counting functions are described by simple functional equations. Therefore the well-developed machinery of analytic combinatorics cannot be applied anymore.

References

[1] A. Bovykin, A. Weiermann, Unprovability, phase transitions and the Riemann zeta-function, in: *New directions in value-distribution theory of zeta and L-functions*, 19–37, Shaker Verlag, Aachen, 2009.

[2] P. Flajolet, R. Selfridge, *Analytic Combinatorics*, Cambridge University Press, Cambridge, 2009.

[3] A. Weiermann, An application of graphical enumeration to PA, *J. Symbolic Logic* **68** (2003), 5–16.

[4] A. Weiermann, Analytic combinatorics for a certain well-ordered class of iterated exponential terms. In: *International Conference on Analysis of Algorithms DMTCS proc. AD*, Vol. 409 (2005), 409–416.

[5] A. Woods, unpublished manuscript