Two positive solutions to a semilinear spectral problem with a convex/concave nonlinearity of variable $q(x)$-power-type ($q(x)(\geq/\leq)1$) and Δ

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This lecture will be posted on the website
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Abstract

We will discuss the question of ***existence*** and ***multiplicity*** of *positive solutions* to the semilinear elliptic Dirichlet problem

$$
(0.1) \qquad \begin{aligned} -\Delta u &= \lambda u + f(x, u(x)) \quad \text{for } x \in \Omega\,; \\ u &= 0 \qquad\qquad\qquad\quad \text{on } \partial\Omega\,, \end{aligned}
$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with the boundary of class $C^{1,\alpha}$, $\lambda \in \mathbb{R}$ a spectral parameter, and $f(x,u) = |u|^{q(x)-1}\,u$ is a **signed $q(x)$-power** of the unknown function (positive variable) $u \in (0,\infty)$ which depends on the point $x \in \Omega$. The Dirichlet Laplace operator being linear, $-\Delta : W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ being an isomorphism, we observe that the elliptic equation in problem (0.1) is ***convex*** (***concave***, respectively) at a given point $x \in \Omega$, provided $q(x) \geq 1$ $(0 < q(x) \leq 1)$. This is a typical problem with variable powers (exponents) treated comprehensively (mostly only for existence) in the monograph by

L. Diening, P. Harjulehto, P. Hästö, and M. Růžička:
"Lebesgue and Sobolev Spaces with Variable Exponents", Lecture Notes in Mathematics, Vol. **2017**, Springer-Verlag, Berlin-Heidelberg, 2011.

In our lecture, besides some necessary attention to the ***existence*** question, we will focus on the existence of ***multiple*** (at least two) positive solutions to problem (0.1) in case the exponent $q(x) - 1$ changes sign inside the domain Ω. The methods we employ have been motivated by the variational and topological arguments used in the work by

P.-L. Lions: *On the existence of positive solutions of semilinear elliptic equations*, SIAM Review, **24**(4), 1982, 441–467.

Similar ideas have been employed simultaneously in the work by H. Brézis, Dj. de Figueiredo, L. Nirenberg, R. E. L. Turner and their joint articles within this group including also P.-L. Lions. Of course, the number of positive (weak) solutions (none or at least one or two) depends on the relation of the spectral parameter $\lambda \in \mathbb{R}$ to some of its critical values.

Finally, if time permits, we will discuss also the classical question of ***uniqueness*** for a related problem with the $p(x)$-Laplacian provided $q(x) \leq \text{const}_1 < \text{const}_2 \leq p(x) - 1$ holds for all $x \in \Omega$. Here, we will show a rather generalized version of the well-known Díaz and Saa inequality in order to derive the uniqueness of a positive (weak) solution. This case corresponds to a ***concave*** nonlinearity above.

Running head: A semilinear elliptic problem with a convex/concave nonlinearity

Keywords: the classical Laplacian; semilinear elliptic Dirichlet problem;
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space-dependent exponent; convex / concave nonlinearity;
positivity and Hopf's boundary point lemma

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