Space-Time Analyticity in the Heston volatility model in Mathematical Finance

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Abstract

We begin by a brief presentation of a well-known mathematical model for European option pricing in a market with stochastic volatility: the popular *Heston volatility model* (Rev. Financial Studies, 1993). European options are used for *market completion*. We explain the connection between a complete market and the analyticity of the weak solution to a general, strongly parabolic linear Cauchy problem of second order in $\mathbb{R}^N \times (0,T)$ $(N = 2)$ with analytic coefficients (in space and time variables). The analytic smoothing property is expressed in terms of holomorphic continuation of global (weak) L^2-type solutions to the system. Given $0 < \xi' < \infty$ and $0 < T' < T < \infty$, we sketch a proof that any L^2-type solution $u : \mathbb{R}^1 \times (0,\infty) \times (0,T) \subset \mathbb{R}^2 \times (0,T) \to \mathbb{R}^1$, $u \equiv u(x,v,t)$, possesses a bounded holomorphic continuation $u(x + \mathrm{i}y, \xi + \mathrm{i}\eta, \sigma + \mathrm{i}\tau)$ into a complex domain in $\mathbb{C}^N \times \mathbb{C}$ $(N = 2)$ defined by $(x,\xi,\sigma) \in \mathbb{R}^1 \times (\xi',\infty) \times (T',T)$, $|y| < A'_1$, $|y| < A'_2$, and $|\tau| < B'$, where $A'_1, A'_2, B' > 0$ are constants depending upon ξ' and T'. The proof uses the extension of a solution to an L^2-type solution in a complex domain in $\mathbb{C}^2 \times \mathbb{C}$, such that this extension satisfies the Cauchy--Riemann equations. The holomorphic extension is thus obtained in a (weighted) Hardy space H^2. A serious difficulty in the Heston model is that the solution is sought only in a half-space $\mathbb{H} = \mathbb{R}^1 \times (0,\infty)$ in $\mathbb{R}^2$ with rather complicated dynamic boundary conditions at the boundary $\partial\mathbb{H} = \mathbb{R}^1 \times \{0\}$; a similarity with the Feller boundary condition (Ann. Math., 1951) will be discussed. We avoid this trouble by a suitable choice of the weight in the weighted L^2 space.

Keywords: Space-time analyticity, parabolic PDE; holomorphic continuation, Hardy space; market completeness, European option

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