Two positive equilibria in a diffusion model with a mixed concave and convex growth in two separated subdomains: an interaction

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Abstract

Population models with a diffusive spread of a population following some kind of nonlinear growth pattern have a long history. Selecting, for the growth, a mixture of a concave nonlinearity at low population densities and a convex one at high densities, in the late 1970s and early 1980s a number of researchers have developed powerful methods of partial differential equations and functional analysis to treat such problems assuming a uniform, space-idependent growth throughout the domain of the habitat. A typical result was the possibility of two positive equilibria of a lower and a higher population density, pointwise ordered throughout the habitat. In our presentation we will construct a similar pair of equilibria in a model with space-dependent growth: concave in one subdomain and convex in the other one, linear on the boundary (or region) between the two subdomains.

We will discuss the question of ***existence*** and ***multiplicity*** of *positive solutions* to the semilinear elliptic Dirichlet problem

$$(0.1) \qquad -\Delta u = \lambda\, u(x)^{q(x)-1} \quad \text{for } x \in \Omega\,; \qquad u = 0 \quad \text{on } \partial\Omega\,,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with the boundary of class $C^{1,\alpha}$, $\lambda \in \mathbb{R}$ a spectral parameter, and $f(x,u) = |u|^{q(x)-1}\, u$ is a **signed $q(x)$-power** of the unknown function of (a positive variable) $u \in (0,\infty)$ which depends on the point $x \in \Omega$.

We will briefly present basic methods for treating the semilinear elliptic Dirichlet problem $\Delta u + \lambda\, u(x)^{q(x)-1} = 0$ with a ***convex*** and ***concave*** nonlinear reaction $f(x,\,\cdot\,) : s \longmapsto |s|^{q(x)-2}s : \mathbb{R}_+ \subset \mathbb{R} \to \mathbb{R}$ which (for $s \geq 0$) is convex in an nonempty open subset $\Omega_+ \stackrel{\text{def}}{=} \{x \in \Omega : q(x) > 2\}$ and concave in another nonempty open subset $\Omega_- \stackrel{\text{def}}{=} \{x \in \Omega : q(x) < 2\}$ of a bounded domain $\Omega \subset \mathbb{R}^N$. Here, $\lambda \in \mathbb{R}_+$ is a nonnegative spectral parameter which decides about the existence and multiplicity of positive weak solutions (at least two). For $0 < \lambda \leq \lambda^*$ we obtain a "stable" C^1-solution $u \equiv u_\lambda$ by monotone iterations. For $0 < \lambda < \lambda^*$ we obtain another C^1-solution, v, by the LERAY–SCHAUDER degree theory, that seems to be "unstable" and satisfies $v(x) > u_\lambda(x)$ for all $x \in \Omega$. Our main tools are fixed points of concave (subhomogeneous) mappings that take advantage of $\Omega_- \neq \emptyset$ in contrast to a priori estimates for "large" positive solutions, v, taking advantage of $\Omega_+ \neq \emptyset$. Our a priori estimates are obtained by a somewhat intriguing application of Young's inequality where Hölder's inequality does not seem to be fine enough. Our main contribution is a method how to handle the interplay between *convex* and *concave* nonlinearities in two disjoint nonempty open subsets of a domain Ω (connected in $\mathbb{R}^N$), as opposed to the classical works assuming a nonlinearity $f(s)$ being *concave* for small values of $s \in \mathbb{R}_+$ and *convex* for large $s \in \mathbb{R}_+$, uniformly in Ω.

The Dirichlet Laplace operator being linear, we observe that the elliptic equation in problem (0.1) is ***convex*** (***concave***, respectively) at a given point $x \in \Omega$, provided $q(x) \geq 2$ $(1 < q(x) \leq 2)$. This is a typical problem with variable powers (exponents).

Finally, if time permits, we will discuss also the classical question of ***uniqueness*** for a related problem with the $p(x)$-Laplacian provided $q(x) \leq \text{const}_1 < \text{const}_2 \leq p(x)$ holds for all $x \in \Omega$.

Running head: A semilinear elliptic problem with a convex/concave nonlinearity

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space-dependent exponent; convex / concave nonlinearity;
positivity and Hopf's boundary point lemma

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