UNIVERSITÉ TOULOUSE 1 CAPITOLE

Toulouse School of Economics

Topics in Mathematics and Economic Decision graduate course

For Masters, MRes, PhD students and researchers

Semi-linear and quasi-linear Black-Scholes-type equations in Mathematical Finance: Analytical and numerical methods with some monotonicity

by **Peter Takáč** (University of Rostock, Germany)

Syllabus:
Whereas *classical linear Black-Scholes* and related parabolic equations have already been used for decades in Mathematical Finance, *non-linear Black-Scholes-type equations* have been introduced into this area much later, perhaps as late as 1998 by G. Barles and H. M. Soner by taking transaction costs in option pricing into account. As this model is one of the favorite ones among those described by *quasi-linear Black-Scholes-type equations*, we will use it as a canonical example in our monotone methods for quasi-linear parabolic problems. The main idea in these quasi-linear Black-Scholes-type equations is the need to introduce and work with ***implied volatility*** as opposed to the ***constant volatility*** used in the classical linear Black-Scholes equations which has proven rather unsuitable especially for longer-time predictions (computations) of option prices even without considering transaction costs. We will briefly mention a “parallel” alternative to models with implied volatility, namely, mode*ls w*ith ***stochastic volatility***, in particular, the ***Heston model*** introduced in 1993 by S. L. Heston. This model is a linear generalization of the Black-Scholes model with an additional “space dimension” for the volatility variable, $\boldsymbol{\xi}$, in addition to the stock price variable, $\mathbf{S}$. Another “parallel” to quasi-linear Black-Scholes-type equations are *semi-linear Black-Scholes-type equations* in counterparty risk models investigated in 2011-13 by Ch. Burgard and M. Kjaer. Although these models consider only constant volatility, they introduce a Lipschitz continuous non-linearity in the **reaction function** in order to account for **credit value adjustments** (CVA). Due to the importance of risk management especially in times of a financial crisis, we will use counterparty risk models to motivate our monotone methods for semi-linear parabolic problems published last year (2022) jointly with Bénédicte Alziary.

We plan to spend one lecture of 90-120 minutes (depending on the patience of the audience) to describe the monotone methods used in the work with Bénédicte Alziary in 2022. These methods take advantage of the ***pointwise order structure*** in, for instance, a Banach space of continuous functions or a Hilbert space of square integrable functions. They are very typical for semi-linear parabolic problems (e.g., reaction-diffusion equations), yielding a monotone increasing (or monotone decreasing) approximation sequence of functions that converge rather fast to the desired solution. In case when rather speed of computation than precision is required, the Monte-Carlo method for solving the linear Black-Scholes-type problem for the successive iteration enjoys significant popularity especially among specialists coming from probability.

This is often the case in the financial industry (investment banks).

I plan to spend three lectures of 90-120 minutes explaining the applications of ***monotone operators*** in Hilbert spaces to solving quasi-linear parabolic problems arising in Black-Scholes-type equations with ***implied volatility***. Until quite recently, these problems were treated as “fully non-linear equations” by the method of *viscosity solutions* (see G. Barles and H. M. Soner in 1998). However, numerous models with implied volatility can be easily transformed (mathematically) into standard quasi-linear parabolic problems with **monotone operators** in a suitable Hilbert space of square integrable functions. The transformation is very easy mathematically with a strong motivation from “practical” finance: One replaces the unknown option price, P, by its partial derivative $\boldsymbol{\Delta} = \partial P / \partial S$ with respect to the stock price, S, called the “Greek Delta” which measures the *sensitivity* of the option price and, thus, plays an important role in the hedging strategy. In the approach by *monotone operators* in a Hilbert space, the “Greek Delta” $\boldsymbol{\Delta} = \boldsymbol{\Delta}(S,t)$ is the unknown function (in place of $P = P(S,t)$) which is calculated directly as the solution of a quasi-linear parabolic problem by the method of *monotone operators*. This approach is motivated by some recent works by A. Bensoussan et al. (2016 and 2022) and V. Barbu et al. (2018 and 2019). They treat particular problems in Stochastic Control and Optimization.

Last but not least, we will discuss numerical methods for our quasi-linear parabolic equations, such es the Faedo-Galerkin approximation of the solution (see, e.g., R. Dautray and J. L. Lions (1985)). We plan to discuss also a numerical method for the linear (but degenerate) Heston equation based on the recent work with Bénédicte Alziary (2020 in arXiv).

Toulouse School of Economics, 1 Esplanade de l’Université - 31080 Toulouse, France