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Semi-linear and quasi-linear Black-Scholes--type equations in Mathematical Finance: Analytical and numerical methods with monotonicity

Lecture 1: Counterparty risk models with semilinear Black-Scholes-type equations

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Risk phenomena and their management have been an important topic of investigation since the financial crisis of 2007 − 2008.
Here, we focus on Counterparty Risk models for options with risky values $V(S,t) \in \mathbb{R}$ modelled by a **semi-linear** Black-Scholes-equation.
A comprehensive monograph:
Andrew Green:
"XVA: Credit, Funding and Capital Valuation Adjustments", John Wiley & Sons Ltd., 2016.

XVA stands for
some ($\equiv$ X) Valuation Adjustment.

Possible values of X are, e.g.,

(1)	$X = C$ − Credit Valuation Adjustment	(CVA)
(2)	$X = D$ − Debit Valuation Adjustment	(DVA)
	to account for **credit risk** C $\rightleftarrows$ D	
(3)	$X = F$ − Funding Valuation Adjustment	(FVA)
(4)	$X = M$ − Margin Valuation Adjustment	(MVA)
(5)	$X = K$ − Capital Valuation Adjustment	(KVA)
(6)	$X = T$ − Tax Valuation Adjustment	(TVA)

The treatment of CVA, DVA, FVA, MVA, KVA and TVA as adjustments reflects the historical development of derivative models and typical bank organisational design rather than the **economic reality** that places credit, funding and capital costs at the centre of accurate pricing and valuation of derivatives.

Since the seminal papers by (**Louis Bachelier** in 1900 at E.N.S. Paris) **Fischer Black and Myron Scholes** and **Robert C. Merton** published in 1973, derivative pricing and valuation has been centred in the Black-Scholes-Merton framework complete with its simplifying assumptions:

- Arbitrage opportunities do not exist.
- Any amount of money can be borrowed or lent at the risk-free rate which is constant and accrues continuously in time (continuously compounded interest rate).
- Any amount of stock can be bought or sold including short selling with no restrictions.
- There are no transaction taxes or margin requirements.
- The underlying asset pays no dividend.
- The asset price is a continuous function with no jumps.
- The underlying asset has a constant volatility.
- Neither counterparty to the transaction is at risk of default.
- *The market is complete*, that is there are no unhedgeable risks.

Often **additional implicit assumptions**
(e.g., the Modigliani-Miller theorem) are imposed
in order to achieve the desired objective(s).

Among the leading researchers in XVA models are
Christoph Burgard (Bank of America – Merrill Lynch)
and **Mats Kjaer** (Bloomberg L.P.);
both earlier at Barclays Capital, London. (Published
after 2011.)

Leading critics of XVA (especially FVA) are
Finance Professors at the University of Toronto,
John Hull and **Alan White**
(published online: 28 Dec 2018).

Mathematics:

Christoph Burgard and **Mats Kjaer** (2011 – 2017).
Let us consider the following **semi-linear**
Black-Scholes-type model with **bilateral CVA**.

1 Introduction

Risk phenomena and their management have been an important topic of investigation since the financial crisis of 2007 – 2008. In this article we focus our attention on ***counterparty risk models*** for options with risky values $\hat{V}(S,t) \in \mathbb{R}$ modelled by a nonlinear Black-Scholes-type equation:

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C)\hat{V} = F(\hat{V}(S,t); S,t) \quad \text{for } (S,t) \in (0,\infty) \times (0,T)\,; \tag{1.1}$$

$$\hat{V}(S,T) = h(S) \quad \text{for } S \in (0,\infty)\,. \tag{1.2}$$

The nonlinearity, $F(\,\cdot\,; S,t) : \mathbb{R} \to \mathbb{R}$, with $\mathbb{R} = (-\infty, +\infty)$ standing for the real line, is given by

$$F(M;S,t) \stackrel{\text{def}}{=} = (R_B\lambda_B + \lambda_C)\, M^- - (\lambda_B + R_C\lambda_C)\, M^+ + s_F\, M^+ \quad \text{for } M \in \mathbb{R} \text{ and } (S,t) \in (0,\infty) \times (0,T)\,, \tag{1.3}$$

where we use the usual abbreviation $x^+ \stackrel{\text{def}}{=} \max\{x,\, 0\}$ and $x^- \stackrel{\text{def}}{=} \max\{-x,\, 0\}$ for $x \in \mathbb{R}$. Hence, $x = x^+ - x^-$ and $|x| = x^+ + x^-$. These kinds of nonlinearities, often called *"jumping nonlinearities*, have a long tradition in Mathematical Modelling.

The parabolic partial differential equation (1.1) (PDE, for short) corresponds to the case when the nonlinearity $F(\,\cdot\,; S,t) : M \mapsto F(M;S,t) : \mathbb{R} \to \mathbb{R}$ on the right-hand side in eq. (1.1) is taken with the ***mark-to-market*** value $M = \hat{V}(S,t)$. This case corresponds to a derivative contract $\hat{V}$ on an asset (stock) $S \in (0,\infty)$ between a ***seller*** **B** and a ***counterparty*** **C** that ***may both default***. The asset price S is not affected by a default of either **B** or **C**, and is assumed to follow the Markov process with the (time-dependent) generator (the Black-Scholes operator) $\mathcal{A}_t$ defined by

$$(\mathcal{A}_t V)(S,t) \stackrel{\text{def}}{=} \frac{1}{2}\,[\sigma(t)]^2 S^2\, \frac{\partial^2 V}{\partial S^2} + [q_S(t) - \gamma_S(t)]S\, \frac{\partial V}{\partial S} \quad \text{for } V : (0,\infty) \times (0,T) \to \mathbb{R} : (S,t) \mapsto V(S,t)\,. \tag{1.4}$$

As usual, we take the volatility, σ, to be a positive constant, $\sigma \in (0,\infty)$. The value of $\gamma_S(t)$ reflects the ***rate of dividend income*** and the value of $q_S(t)$ is the ***net share position financing cost*** which depends on the ***risk-free rate*** $r(t)$ and the ***repo-rate*** of $S(t)$. "Typical" values for the terminal condition (1.2) are $h(S) \equiv V_T(S)$ where $V_T(S) = (S-K)^+ = (\mathrm{e}^X - K)^+$ for $X = \log\, S \in \mathbb{R}$ (in case of the *European call option*) and $V_T(S) = (S-K)^- = (K-S)^+ = (K - \mathrm{e}^X)^+$ (for the *European put option*).

The simple transformation $S \mapsto X = \log\, S : (0,\infty) \to \mathbb{R}$ of the ***asset (stock) price*** $S \in (0,\infty)$ into the ***logarithmic asset (stock) price*** $X = \log\, S \in \mathbb{R}$ has a "mathematical" justification in transforming the degenerate elliptic differential operator $\mathcal{A}_t$ in formula (1.4) into the regular differential operator $\mathcal{A}(\tau)$ in formula (2.3) below. Further connections between the asset price S and the logarithmic asset price $X = \log\, S$ can be found in the monograph by J.-P. FOUQUE, G. PAPANICOLAOU, R. SIRCAR, and K. SØLNA [20, Sect. 1].

A frequently used *alternative* to our choice $M = \hat{V}(S,t)$ of the ***mark-to-market*** value M in the nonlinearity $F(M; S, t)$ on the right-hand side in eq. (1.1) is $M = V(S,t)$ where V denotes the same derivative between two parties that ***cannot default***; see e.g. F. BAUSTIAN, M. FENCL, J. POSPÍŠIL, and V. ŠVÍGLER [6, Sect. 2] for numerical treatment. This ***risk--free value***, V, satisfies the classical (linear) Black-Scholes PDE (partial differential equation) with the prescribed terminal value $V(S,T) = h(S)$ for $S \in (0,\infty)$ at ***maturity time*** $t = T$. Inserting this known value $F(V(S,t); S, t)$ in eq. (1.1) in place of $F(\hat{V}(S,t); S, t)$, we thus obtain an inhomogeneous linear equation for another (new) value of $\hat{V}(S,t)$. We refer to the works by C. BURGARD and M. KJAER [12], [13, Section 3], and [14] for details concerning modelling and to I. ARREGUI, B. SALVADOR, and C. VÁZQUEZ [3] for numerical results. We warn the reader that Refs. [3] and [12, 13, 14] use the convention $V = V^+ + V^-$ with $V^+ \stackrel{\text{def}}{=} \max\{V, 0\}$ and $V^- \stackrel{\text{def}}{=} \min\{V, 0\}$ (≤ 0) for $V \in \mathbb{R}$; nevertheless, we will stick with our notation $V = V^+ - V^-$ with $V^- \stackrel{\text{def}}{=} \max\{-V, 0\}$ (≥ 0). We will not worry about this alternative any more and focus entirely on the nonlinear equation (1.1). Making use of eq. (1.3), we arrive at the following equivalent form of eq. (1.1), frequently used, cf. [13, Section 2, Eq. (1)]:

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = -(1-R_B)\lambda_B \hat{V}^- + (1-R_C)\lambda_C \hat{V}^+ + s_F \hat{V}^+ \quad \text{for } (S,t) \in (0,\infty) \times (0,T)\,. \tag{1.5}$$

This backward parabolic equation is supplemented by the ***terminal condition*** (1.2).

Models with ***nonlinearities*** are neither popular nor very frequent in Mathematical Finance. In the present article we treat a class of semilinear parabolic equations of type (1.5) with the standard linear diffusion operator $\frac{\partial}{\partial t} + \mathcal{A}_t$ and the nonlinear reaction function that is more general that the one on the right-hand side of (1.5) (only uniformly Lipschitz-continuous). As far as we know, this class was introduced in the work by C. BURGARD and M. KJAER [13, Section 3] and [14]. Another class of nonlinear models is based on a ***nonlinear Black-Scholes PDE*** with the quasilinear diffusion operator $\frac{\partial \hat{V}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \dots$, where the volatility $\sigma \equiv \sigma\big(\frac{\partial^2 V}{\partial S^2}\big)$ depends on the second partial derivative, and with a "typical" linear reaction function (sometimes including also *transaction costs*). This class can be traced to G. BARLES and H. M. SONER [5, Eq. (1.2), p. 372] with some additional analytic studies (on explicit solutions) performed in L. A. BORDAG and Y. CHMAKOVA [9]. Some additional references to related numerical studies and simulations will be added in Sections 4 and 5.

Last but not least, we would like to mention *"modelling of incertitude in the environment"* investigated in the works by G. DÍAZ, J. I. DÍAZ, and CH. FAGHLOUMI [16] and J. I. DÍAZ and CH. FAGHLOUMI [15, 17]. The stochastic formulation in these problems leads to *degenerate obstacle problems* closely related to parabolic problems with a free boundary that arise in *Black--Scholes PDEs* for ***American options***. We believe that our current developments of D. H. SATTINGER's *monotone methods* [42] are applicable also to these kinds of Black-Scholes PDEs with a *free boundary*.

This article is organized as follows. We begin with a functional-analytic reformulation of the B-S equation (1.5) in the next section (Section 2). The terminal value problem (1.5), (1.2) will be transformed into an initial value Cauchy problem of parabolic type. This Cauchy problem is an initial value problem for the nonlinear (semilinear) B-S equation with a uniformly Lipschitz--continuous (nonlinear) reaction function, as well. In Section 3 we construct a ***monotone iteration scheme*** of *supersolutions* of this B-S equation that converge as a monotone decreasing (i.e., nonincreasing) sequence to the solution from above; see our main result, Theorem 3.4. A closely related ramification of this monotone iteration scheme provides an increasing sequence of *subsolutions* of the B-S equation that converge to the solution from below; see Remark 3.5.

Numerical methods play an important role in Mathematical Finance. In Section 4 we discuss applications of two most common methods to Mathematical Finance, *finite differences/elements* and *Monte Carlo*. We discuss their advantages and problems when compared to each other. Finally, in Section 5 we derive an *explicit formula* for the solution of the inhomogeneous linear parabolic initial value problem for the B-S equation that serves for computing the monotone iteration scheme in Section 3. This formula is obtained by ***variation-of-constants*** (with integrals over $\mathbb{R}^1$ and $[0,T]$) which makes it interesting for *Monte Carlo* computations. On the other hand, the solution of the inhomogeneous linear parabolic problem can be computed also by *finite differences/elements*.

2 Functional-analytic reformulation of the B-S equation

We wish to treat the *terminal value problem* (1.5) (or, equivalently, (1.1)) above, with the terminal condition (1.2), by standard analytic and numerical methods for semilinear parabolic *initial value problems*. To this end, we rewrite problem (1.5), (1.2) as the following general initial value problem for the unknown function $v : \mathbb{R}^1 \times (0,T) \to \mathbb{R}$,

$$\frac{\partial v}{\partial \tau} - \mathcal{A}(\tau) v + r\, v = \tilde{F}(v(x,\tau); x, \tau) \quad \text{for } (x,\tau) \in \mathbb{R}^1 \times (0,T)\,; \tag{2.1}$$

$$v(x,0) = v_0(x) \stackrel{\text{def}}{=} h(\mathrm{e}^x) \quad \text{for } x \in \mathbb{R}^1\,, \tag{2.2}$$

where $\mathcal{A}(\tau)$ denotes the ***Black-Scholes operator*** defined by

$$\begin{aligned} &(\mathcal{A}(\tau)v)(x,\tau) \stackrel{\text{def}}{=} (\mathcal{A}_{T-\tau}v)(x,\tau) \\ &= \frac{1}{2}\,[\sigma(T-\tau)]^2\,\frac{\partial^2 v}{\partial x^2} + \left(q_S(T-\tau) - \gamma_S(T-\tau) - \frac{1}{2}\,[\sigma(T-\tau)]^2 \right) \frac{\partial v}{\partial x} \\ &\qquad \text{for } v : \mathbb{R}^1 \times (0,T) \to \mathbb{R} : (x,\tau) \mapsto v(x,\tau)\,, \end{aligned} \tag{2.3}$$

and the nonlinearity $\tilde{F}(\,\cdot\,; x,\tau) : \mathbb{R} \to \mathbb{R}$ is given by

$$\begin{aligned} \tilde{F}(v;x,\tau) &\stackrel{\text{def}}{=} -F(v;\mathrm{e}^x, T-\tau) - (\lambda_B + \lambda_C)\, v \\ &= (1-R_B)\lambda_B\, v^- - (1-R_C)\lambda_C\, v^+ - s_F\, v^+ \\ &\qquad \text{for } v \in \mathbb{R} \text{ and } (x,\tau) \in \mathbb{R}^1 \times (0,T)\,. \end{aligned} \tag{2.4}$$

Here, $\tau = T - t$ stands for the ***time to maturity*** and $x = \log\ S$ is the ***logarithmic asset (stock) price***; we take $(x,\tau) \in \mathbb{R}^1 \times (0,T)$. In the sequel we will never use the ***real time***

$t = T - \tau \in (0,T)$ any more, so we prefer to use the letter t in place of τ to denote the ***time to maturity***, as it is usual in parabolic problems. According to this new notation, in eq. (2.3) we replace the time-dependent coefficients $\sigma(T-\tau)$, $q_S(T-\tau)$, and $\gamma_S(T-\tau)$ by $\sigma(t)$, $q_S(t)$, and $\gamma_S(t)$, respectively, and thus forget about the original terminal value problem (1.5), (1.2):

$$(2.5)\qquad (\mathcal{A}(t)v)(x,t) \stackrel{\text{def}}{=} \frac{\partial}{\partial x}\left[\frac{1}{2}[\sigma(t)]^2\frac{\partial v}{\partial x} + \left(q_S(t) - \gamma_S(t) - \frac{1}{2}[\sigma(t)]^2\right)v(x,t)\right]$$
$$\text{for } \ v : \mathbb{R}^1 \times (0,T) \to \mathbb{R} : \ (x,t) \mapsto v(x,t)\,.$$

Next, in order to make the initial value problem (2.1), (2.2) compatible with the ***monotone methods*** described in the article by DAVID H. SATTINGER [42], we rewrite this problem as follows:

$$(2.6)\qquad \frac{\partial v}{\partial t} - \mathcal{A}(t)v + (r + L_{\tilde F})\,v = \tilde F(v(x,t);x,t) + L_{\tilde F}\,v \quad \text{ for } \ (x,t) \in \mathbb{R}^1 \times (0,T)\,;$$

with the initial condition $v(x,0) = v_0(x) \stackrel{\text{def}}{=} h(\mathrm{e}^x)$ for $x \in \mathbb{R}^1$ in eq. (2.2), where the constant $L_{\tilde F} \in \mathbb{R}_+ = [0,\infty)$ is defined by

$$(2.7)\qquad L_{\tilde F} = \max\{(1-R_B)\lambda_B,\ (1-R_C)\lambda_C + s_F\}\,.$$

According to [13], $s_F \equiv r_F - r$, $\lambda_B \equiv r_B - r$, and $\lambda_C \equiv r_C - r$ are some nonnegative constants and $R_B, R_C \in [0,1]$ are the ***recovery rates*** on the derivative positions of parties B and C, respectively. As a consequence, the function $G(\,\cdot\,;x,t) : v \mapsto G(v;x,t) : \mathbb{R} \to \mathbb{R}$, defined by

$$(2.8)\qquad \begin{aligned} G(v;x,t) &= \tilde F(v;x,t) + L_{\tilde F}\,v \\ &= -\left[L_{\tilde F} - (1-R_B)\lambda_B\right]v^- + \left[L_{\tilde F} - (1-R_C)\lambda_C - s_F\right]v^+ \\ &\qquad\qquad \text{for } v \in \mathbb{R} \text{ and } \ (x,t) \in \mathbb{R}^1 \times (0,T)\,, \end{aligned}$$

is monotone increasing (i.e., nondecreasing) on $\mathbb{R}$. Notice that both functions, $v \mapsto -v^-$ and $v \mapsto v^+$, are nondecreasing on $\mathbb{R}$. Indeed, we have also

$$\frac{\partial G}{\partial v}(v;x,t) = \frac{\partial \tilde F}{\partial v} + L_{\tilde F} = \begin{cases} L_{\tilde F} - (1-R_B)\lambda_B & \text{if } \ v < 0\,, \\ L_{\tilde F} - ((1-R_C)\lambda_C + s_F) & \text{if } \ v > 0\,; \end{cases}$$
$$\text{with} \quad 0 \le \frac{\partial G}{\partial v}(v;x,t) \le L_{\tilde F} \quad \text{ for all } \ v \in \mathbb{R}^1 \setminus \{0\} \ \text{ and } \ (x,t) \in \mathbb{R} \times (0,T)\,.$$

In addition, the left-hand side of eq. (2.6) clearly satisfies the weak maximum principle.

An alternative to D. H. SATTINGER's article [42] on *monotone methods* for "Nonlinear Elliptic and Parabolic Problems" is offered in a book form in the monograph C. V. PAO [39, Chapt. 1], §1.5 (pp. 20–26) and §1.7 (pp. 31–36), and further in Chapters 2 and 3. Numerous well-known details are included here, such as the relation between the ***one-sided Lipschitz condition*** expressed in the inequality on the left-hand side,

$$(2.9)\qquad -L_{\tilde F} \le \frac{\partial \tilde F}{\partial v} \le L_{\tilde F} \quad \text{ for all } \ v \in \mathbb{R}^1 \setminus \{0\} \ \text{ and } \ (x,t) \in \mathbb{R} \times (0,T)\,,$$

and the ***monotonicity hypothesis*** (**G3**) below.

We now specify our hypotheses on the general nonlinearity $G : \mathbb{R} \times \mathbb{R}^1 \times (0,T)$ on the right-hand side of eq. (2.6) treated in our present work. We assume that G satisfies the following hypotheses:

Hypotheses (G1) For each fixed $v \in \mathbb{R}$, the function
$G(v;\,\cdot\,,\,\cdot\,) : (x,t) \mapsto G(v;x,t) : \mathbb{R}^1 \times (0,T) \to \mathbb{R}$ is Lebesgue-measurable.

(G2) For almost every pair $(x,t) \in \mathbb{R}^1 \times (0,T)$, the function $G(\,\cdot\,;x,t) : v \mapsto G(v;x,t) : \mathbb{R} \to \mathbb{R}$ is uniformly Lipschitz-continuous with a Lipschitz constant $L_G \in \mathbb{R}_+$, that is, we have

$$\begin{aligned} &|G(v_1;x,t) - G(v_2;x,t)| \leq L_G\,|v_1 - v_2| \\ &\quad \text{for all } \; v_1, v_2 \in \mathbb{R} \; \text{ and for almost all } \; (x,t) \in \mathbb{R}^1 \times (0,T)\,. \end{aligned} \tag{2.10}$$

(G3) For almost every pair $(x,t) \in \mathbb{R}^1 \times (0,T)$, the function $G(\,\cdot\,;x,t) : v \mapsto G(v;x,t) : \mathbb{R} \to \mathbb{R}$ is monotone increasing, that is, $v_1 \leq v_2$ in $\mathbb{R}$ implies $G(v_1;x,t) \leq G(v_2;x,t)$.

(G4) There is a constant $C_0 \in \mathbb{R}_+$ such that, at almost every time $t \in (0,T)$, the function $G(0;\,\cdot\,,t) : x \mapsto G(0;x,t) : \mathbb{R}^1 \to \mathbb{R}$ satisfies the *exponential growth* restriction

$$|G(0;x,t)| \leq C_0 \cdot \exp(|x|) \quad \big(\leq C_0(\mathrm{e}^{x} + \mathrm{e}^{-x}) \big) \qquad \text{for almost all } \; x \in \mathbb{R}^1\,. \tag{2.11}$$

(G5) There are constants $C_1 \in \mathbb{R}_+$ and $\vartheta_G \in (0,1)$ such that, for every $v \in \mathbb{R}$ and almost every $x \in \mathbb{R}^1$, the function $G(v;x,\,\cdot\,) : t \mapsto G(v;x,t) : (0,T) \to \mathbb{R}$ is Hölder-continuous with the *Hölder exponent* ϑ_G $(0 < \vartheta_G < 1)$ in the following sense,

$$\begin{aligned} &|G(v;x,t_1) - G(v;x,t_2)| \leq C_1\,|v| \cdot |t_1 - t_2|^{\vartheta_G} \\ &\quad \text{for all } \; t_1, t_2 \in (0,T) \; \text{ and for almost all } \; (v,x) \in \mathbb{R} \times \mathbb{R}^1\,. \end{aligned} \tag{2.12}$$

From now on, let us consider the following generalization of the initial value problem (2.6), (2.2):

$$\frac{\partial v}{\partial t} - \mathcal{A}(t)v + r_G\,v = G(v(x,t);x,t) \qquad \text{for } \; (x,t) \in \mathbb{R}^1 \times (0,T)\,; \tag{2.13}$$

with the initial condition $v(x,0) = v_0(x) \stackrel{\text{def}}{=} h(\mathrm{e}^x)$ for $x \in \mathbb{R}^1$ in eq. (2.2), where the constant $r + L_{\tilde{F}}$ in eq. (2.6) has been replaced by the new constant $r_G \in \mathbb{R}_+$, owing to our monotonicity hypothesis (**G3**). Concerning hypotheses on the time-dependent coefficients that appear in the Black-Scholes operator $\mathcal{A}(t)$ defined in eq. (2.5) (recall that $\tau = t$), we assume the following Hölder continuity:

Hypotheses

(BS1) $\sigma : [0,T] \to (0,\infty)$ is a positive, Hölder-continuous function satisfying

$$|\sigma(t_1) - \sigma(t_2)| \leq C_\sigma \, |t_1 - t_2|^{\vartheta_\sigma} \quad \text{for all } \, t_1, t_2 \in [0,T] \,, \tag{2.14}$$

where $C_\sigma \in \mathbb{R}_+$ and $\vartheta_\sigma \in (0,1)$ are some constants independent from time $t \in [0,T]$.

(BS2) $q_S, \gamma_S : [0,T] \to \mathbb{R}$ is a pair of Hölder-continuous function satisfying

$$|q_S(t_1) - q_S(t_2)| \leq C_q \, |t_1 - t_2|^{\vartheta_q} \quad \text{and} \tag{2.15}$$

$$|\gamma_S(t_1) - \gamma_S(t_2)| \leq C_\gamma \, |t_1 - t_2|^{\vartheta_\gamma} \quad \text{for all } \, t_1, t_2 \in [0,T] \,, \tag{2.16}$$

where $C_q, C_\gamma \in \mathbb{R}_+$ and $\vartheta_q, \vartheta_\gamma \in (0,1)$ are some constants (independent from $t \in [0,T]$).

Remark 2.1 (Hölder exponents.) In Hypotheses (**G5**), (**BS1**), and (**BS2**) we may and will replace the *Hölder exponents* ϑ_G, ϑ_σ, ϑ_q, and ϑ_γ by their minimum ϑ_0,

$$\vartheta_0 = \min\{\vartheta_G, \, \vartheta_\sigma, \, \vartheta_q, \, \vartheta_\gamma\} \,, \quad \vartheta_0 \in (0,1) \,.$$

□

Clearly, from Hypothesis (**BS1**) we derive $\sigma(t) \geq \sigma_0 = \min_{t\in[0,T]} \sigma(t) > 0$ for all $t \in [0,T]$. This fact, combined with (**BS2**), guarantees the uniform ellipticity of the Black-Scholes operator $\mathcal{A}(t)$ independently from $t \in [0,T]$.

Remark 2.2 (Risk-free interest rate.) One may also suggest to replace the multiplicative constant $r_G \in \mathbb{R}$ on the left-hand side of eq. (2.13) by the time-dependent ***risk-free interest rate*** $r : [0,T] \to \mathbb{R}$ satisfying a Hölder continuity condition analogous to those in eqs. (2.15) and (2.16). However, this change would not make eq. (2.13) more general in that it could be reduced to the present form (2.13) with the term $r_G \, v$ as follows:

First, define $r_G \in \mathbb{R}$ by $r_G = \max_{t\in[0,T]} r(t)$; then replace the function $G(v;x,t)$ on the right-hand side of eq. (2.13) by the sum $G_r(v;x,t) = G(v;x,t) + [r_G - r(t)]\, v$ for $(v;x,t) \in \mathbb{R} \times \mathbb{R}^1 \times (0,T)$. Clearly, thanks to $r_G - r(t) \geq 0$ for every $t \in [0,T]$, the function $G_r(\,\cdot\,;\,\cdot\,,\,\cdot\,) : (v;x,t) \mapsto G_r(v;x,t) : \mathbb{R}^1 \times (0,T) \to \mathbb{R}$ satisfies all Hypotheses (**G1**) – (**G5**) imposed on the function G. We conclude that the *interest rate difference*, $r_G - r(t)$, can be included in the reaction function G. We thus keep eq. (2.13) in the present form with $r_G \in \mathbb{R}$ being a given constant. □

Our last hypothesis in problem (2.13), (2.2) restricts the growth of the initial condition $v(x,0) = v_0(x) \stackrel{\text{def}}{=} h(\mathrm{e}^x)$ for $x \in \mathbb{R}^1$ in eq. (2.2) as follows:

Hypothesis

($\mathbf{v_0}$) The function $v_0 : \mathbb{R} \to \mathbb{R}$ is Lebesgue-measurable and there is a constant $C_h \in \mathbb{R}_+$ such that, for almost all $x \in \mathbb{R}^1$, we have

$$|v_0(x)| = |h(\mathrm{e}^x)| \leq C_h \cdot \exp(|x|) \quad \big(\leq C_h(\mathrm{e}^x + \mathrm{e}^{-x}) \big) \,. \tag{2.17}$$

As we have already indicated in our hypothesis (**G4**) on the *exponential growth* restriction of G, we are going to look for (strong, weak or mild) solutions $v : \mathbb{R}^1 \times (0,T) \to \mathbb{R}$ to the initial value problem (2.13), (2.2) satisfying an analogous *exponential growth* restriction of type $v(\,\cdot\,,t) \in H_{\mathbb{C}}$ at every time $t \in (0,T)$, where $H_{\mathbb{C}} = L^2(\mathbb{R};\mathfrak{w})$ denotes the *complex* Hilbert space of all complex-valued Lebesgue-measurable functions $f : \mathbb{R} \to \mathbb{C}$ with the finite norm

$$\|f\|_H \stackrel{\text{def}}{=} \left(\int_{\mathbb{R}} |f(x)|^2\, \mathfrak{w}(x)\,\mathrm{d}x\right)^{1/2} < \infty\,,$$

where $\mathfrak{w}(x) \stackrel{\text{def}}{=} \mathrm{e}^{-\mu|x|}$ is a weight function with some constant $\mu \in (2,\infty)$. This norm is induced by the inner product

$$(f,g)_H \equiv (f,g)_{L^2(\mathbb{R};\mathfrak{w})} \stackrel{\text{def}}{=} \int_{\mathbb{R}} f\,\bar{g}\cdot\mathfrak{w}(x)\,\mathrm{d}x \quad \text{ for } f,g \in H_{\mathbb{C}}\,.$$

As usual, the symbol $\bar{z}$ denotes the complex conjugate of a complex number $z \in \mathbb{C}$ where $\mathbb{C} = \mathbb{R} + \mathrm{i}\mathbb{R}$ is the complex plane. We consider the complex Hilbert space $H_{\mathbb{C}}$ only for better understanding of our applications using ***holomorphic semigroups*** in $H_{\mathbb{C}}$ generated by the (unbounded) *Black-Scholes operator* $\mathcal{A}(t) : H_{\mathbb{C}} \to H_{\mathbb{C}}$ in eq. (2.13) above. Our solutions $v(x,t)$ to the initial value problem (2.13), (2.2) will be always real-valued, i.e., $v(\,\cdot\,,t) \in H$ at every time $t \in (0,T)$, where H denotes the closed *real* vector subspace of all real-valued functions $f : \mathbb{R} \to \mathbb{R}$ from $H_{\mathbb{C}}$. The domain of the differential operator $\mathcal{A}(t)$, denoted by $\mathcal{D}(\mathcal{A}(t))$, is a *complex* vector subspace of $H_{\mathbb{C}}$ which is independent from time $t \in [0,T]$, i.e., $\mathcal{D}(\mathcal{A}(t)) \equiv D_{\mathbb{C}} \subset H_{\mathbb{C}}$ for every $t \in [0,T]$. The vector space $D_{\mathbb{C}}$ consists of all functions $f \in H_{\mathbb{C}}$ whose weak (distributional) derivatives $f' = \frac{\mathrm{d}f}{\mathrm{d}x}$ and $f'' = \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$ belong to $H_{\mathbb{C}}$, as well. We set $D = D_{\mathbb{C}} \cap H$ to denote the closed *real* vector subspace of all real-valued functions $f : \mathbb{R} \to \mathbb{R}$ from $D_{\mathbb{C}}$. The vector space $D_{\mathbb{C}}$ becomes a Banach space under the norm

$$\|f\|_D \stackrel{\text{def}}{=} \|f\|_H + \|f''\|_H \quad \text{ for } f \in D_{\mathbb{C}}\,.$$

This norm is equivalent with the stronger norm

$$|\!|\!|f|\!|\!|_D \stackrel{\text{def}}{=} \|f\|_H + \|f'\|_H + \|f''\|_H \quad \text{ for } f \in D_{\mathbb{C}}\,,$$

by a simple interpolation inequality. We refer to the monograph by K.-J. Engel and R. Nagel [18] for details concerning ***holomorphic semigroups*** and their (infinitesimal) generators, especially to [18, Chapt. II, Sect. 4*a*, pp. 96–109].

We denote by $H^1_{\mathbb{C}}$ the complex interpolation space between $D_{\mathbb{C}}$ and $H_{\mathbb{C}}$ that consist of all functions $f : \mathbb{R} \to \mathbb{R}$ from $H_{\mathbb{C}}$ such that both $f, f' \in H_{\mathbb{C}}$. $H^1_{\mathbb{C}}$ is a vector space which becomes a Banach space under the norm

$$\|f\|_{H^1} \stackrel{\text{def}}{=} \|f\|_H + \|f'\|_H \quad \text{ for } f \in H^1_{\mathbb{C}}\,.$$

Hence, we have the continuous imbeddings $D_{\mathbb{C}} \hookrightarrow H^1_{\mathbb{C}} \hookrightarrow H_{\mathbb{C}}$. Moreover, given any fixed $t \in [0,T]$, $H^1_{\mathbb{C}}$ is the domain of the *sesquilinear form*

$$
\begin{aligned}
&\mathcal{Q}(t) : H^1_{\mathbb{C}} \times H^1_{\mathbb{C}} \to \mathbb{C} : (f,g) \longmapsto \mathcal{Q}(t)(f,g) \stackrel{\mathrm{def}}{=} - \int_{\mathbb{R}} [\mathcal{A}(t)f](x)\, \overline{g(x)} \cdot \mathfrak{w}(x)\, \mathrm{d}x \\
&= - \int_{-\infty}^{+\infty} \left\{ \tfrac{1}{2} [\sigma(t)]^2 f''(x) + \left(q_S(t) - \gamma_S(t) - \tfrac{1}{2} [\sigma(t)]^2 \right) f'(x) \right\} \overline{g(x)} \cdot \mathfrak{w}(x)\, \mathrm{d}x \\
&= \int_{-\infty}^{+\infty} \Big\{ \tfrac{1}{2} [\sigma(t)]^2 f'(x)\, \overline{g'(x)} - \tfrac{\mu}{2} [\sigma(t)]^2 \operatorname{sign}(x) f'(x)\, \overline{g(x)} \\
&\qquad - \left(q_S(t) - \gamma_S(t) - \tfrac{1}{2} [\sigma(t)]^2 \right) f'(x)\, \overline{g(x)} \Big\} \cdot \mathfrak{w}(x)\, \mathrm{d}x \\
&= \tfrac{1}{2} [\sigma(t)]^2 \cdot \textstyle\int_{-\infty}^{+\infty} f'(x)\, \overline{g'(x)} \cdot \mathfrak{w}(x)\, \mathrm{d}x \\
&+ \tfrac{\mu}{2} [\sigma(t)]^2 \cdot \left(\textstyle\int_{-\infty}^{0} - \int_{0}^{+\infty} \right) f'(x)\, \overline{g(x)} \cdot \mathfrak{w}(x)\, \mathrm{d}x \\
&- \left(q_S(t) - \gamma_S(t) - \tfrac{1}{2} [\sigma(t)]^2 \right) \textstyle\int_{-\infty}^{+\infty} f'(x)\, \overline{g(x)} \cdot \mathfrak{w}(x)\, \mathrm{d}x \\
&\qquad \text{defined first only for } f,g \in D_{\mathbb{C}}\,.
\end{aligned}
\tag{2.18}
$$

The continuous extension of $\mathcal{Q}(t)(f,g)$ to all $f,g \in H^1_{\mathbb{C}}$ is immediate, thanks to $D_{\mathbb{C}}$ being a dense vector subspace of $H^1_{\mathbb{C}}$. A few simple applications of the Cauchy-Schwartz inequality show that the (non-symmetric) *sesquilinear form* $\mathcal{Q}(t)$ on $H^1_{\mathbb{C}}$ is ***coercive***. Indeed, with a help from Hypothesis (**BS1**) we have $\sigma(t) \geq \sigma_0 = \min_{t\in[0,T]} \sigma(t) > 0$ for all $t \in [0,T]$. Consequently, if the constant $\lambda_0 \in (0,\infty)$ below is chosen sufficiently large, then we get the following more precise quantification of coercivity at every time $t \in [0,T]$:

$$
\mathcal{Q}(t)(f,f) + \lambda\, (f,f)_H \geq \frac{\sigma_0}{4}\, \|f\|^2_{H^1} + \|f\|^2_H \quad \text{for every } \lambda \geq \lambda_0\,,
\tag{2.19}
$$

thanks to Hypotheses (**BS1**) and (**BS2**). We note that the constant λ_0 depends neither on time $t \in [0,T]$ nor on the number $\lambda \geq \lambda_0$.

Let $I \equiv I_H$ denote the identity mapping on $H_{\mathbb{C}}$. Given any real number $\lambda \geq \lambda_0$, from ineq. (2.19) we infer that the linear operator

$$
- \mathcal{A}_\lambda(t) \stackrel{\mathrm{def}}{=} - \mathcal{A}(t) + \lambda I : D_{\mathbb{C}} \subset H_{\mathbb{C}} \to H_{\mathbb{C}}
$$

is an isomorphism of the Banach space $D_{\mathbb{C}}$ onto another Banach space $H_{\mathbb{C}}$, both, algebraically and topologically. Now we can apply the well-known results for abstract linear initial value problems of parabolic type, e.g., from L. C. EVANS [19], Chapt. 7, §1.1, p. 352, or J.-L. LIONS [35], Chapt. IV, §1, p. 44, or [36], Chapt. III, eq. (1.11), p. 102, to conclude that the inhomogeneous linear parabolic initial value problem

$$
\frac{\partial v}{\partial t} - \mathcal{A}(t) v + r_G\, v = f(x,t) \quad \text{for } (x,t) \in \mathbb{R}^1 \times (0,T)\,;
\tag{2.20}
$$

with the initial condition $v(x,0) = v_0(x) \stackrel{\mathrm{def}}{=} h(\mathrm{e}^x)$ for $x \in \mathbb{R}^1$ in eq. (2.2), possesses a unique ***weak solution*** $v : [0,T] \to H$, whenever the initial value $v_0 \in H$ is given, such that $v_0 : \mathbb{R}^1 \to \mathbb{R}$ obeys ineq. (2.17) in Hypothesis ($\mathbf{v_0}$). The weak solution v is continuous as an H-valued function of time $t \in [0,T]$, that is, $v \in C([0,T] \to H)$. This is the "linear part" of the semilinear problem (2.13), (2.2) with a prescribed inhomogeneity $f : [0,T] \to H$ that is assumed to be *strongly Lebesgue-measurable* and (essentially) bounded on $(0,T)$, i.e., $f \in L^\infty((0,T) \to H)$. In our case, $f \in L^\infty((0,T) \to H)$ follows from our stronger hypothesis below:

Hypothesis

(f1) $f : [0,T] \to H \subset H_{\mathbb{C}} = L^2(\mathbb{R}; \mathfrak{w})$ is a continuous function, i.e., $f \in C([0,T] \to H)$, where $f : \mathbb{R}^1 \times [0,T] \to \mathbb{R} : (x,t) \mapsto f(x,t)$ satisfies $f(t) \equiv f(\,\cdot\,,t) \in H$ for every $t \in [0,T]$.

Although in the following sections we work only with weak solutions $v : [0,T] \to H$, $v(0) = v_0 \in H$, to the "linear part" (2.20) of the semilinear equation (2.13), we would like to remark that the unique ***weak solution*** $v : [0,T] \to H$ to the linear initial value problem (2.20), (2.2) becomes a (unique) ***classical solution*** if $f : [0,T] \to H$ satisfies the following stronger, Hölder-continuity hypothesis with the *Hölder exponent* $\vartheta_f \in (0,1)$:

Hypothesis

(f1') $f : [0,T] \to H \subset H_{\mathbb{C}} = L^2(\mathbb{R}; \mathfrak{w})$ is a ϑ_f-Hölder-continuous function, i.e., there are constants $\vartheta_f \in (0,1)$ and $C_f \in \mathbb{R}_+$ such that

$$\|f(\,\cdot\,,t_1) - f(\,\cdot\,,t_2)\|_H \le C_f \cdot |t_1 - t_2|^{\vartheta_f} \quad \text{for all } t_1, t_2 \in [0,T]\,. \tag{2.21}$$

This is the case if the inhomogeneity $f : \mathbb{R}^1 \times [0,T] \to \mathbb{R}$ satisfies $f(\,\cdot\,,t) \in H$ for each $t \in [0,T]$ and there are some constants $\tilde{C}_f \in \mathbb{R}_+$ and $\kappa \in \mathbb{R}$, with $1 \le \kappa < \mu/2$, such that

$$\begin{gathered} |f(x,t_1) - f(x,t_2)| \le \tilde{C}_f \, \mathrm{e}^{\kappa|x|} \cdot |t_1 - t_2|^{\vartheta_f} \\ \text{for a.e. (almost every) } x \in \mathbb{R}^1 \text{ and for all } t_1, t_2 \in [0,T]\,. \end{gathered} \tag{2.22}$$

Recall that μ ($\mu > 2$) is the constant in the weight function $\mathfrak{w}(x) \stackrel{\text{def}}{=} \mathrm{e}^{-\mu|x|}$ in the Hilbert space $H_{\mathbb{C}} = L^2(\mathbb{R}; \mathfrak{w})$. We remark that for the inhomogeneous linear equation (2.20), the nonlinearity G on the right-hand side of eq. (2.13) becomes $G(v; x, t) \equiv f(x,t)$; hence, we have $\vartheta_G = \vartheta_f$ in Hypothesis (**G5**). Let us recall that, by Remark 2.1, we have replaced the *Hölder exponents* ϑ_G, ϑ_σ, ϑ_q, and ϑ_γ by their minimum ϑ_0; hence, we may include also the value of ϑ_f in that minimum:

$$\vartheta_0 = \min\{\vartheta_G,\, \vartheta_\sigma,\, \vartheta_q,\, \vartheta_\gamma,\, \vartheta_f\}\,, \quad \vartheta_0 \in (0,1)\,. \tag{2.23}$$

Indeed, according to the existence, uniqueness, and regularity results for problem (2.20), (2.2) in A. PAZY [40, Chapt. 5, §5.7], Theorem 7.1 on p. 168, if (2.21) holds, then the unique ***weak solution*** $v : [0,T] \to H$ to the linear initial value problem (2.20), (2.2) described above happens to be a unique ***classical solution*** which, among other properties, is continuous as a function $v : [0,T] \to H$, i.e., $v \in C([0,T] \to H)$, continuously differentiable on the time interval

$(0,T]$, $v(t) \equiv v(\,\cdot\,,t) \in D$ for every $t \in (0,T]$, i.e., $\mathcal{A}(t)v(t) \in H$ for $t \in (0,T]$, and v satisfies the abstract differential equation

$$\frac{\partial v}{\partial t} - \mathcal{A}(t)v + r_G\, v = f(t) \qquad \text{in } H \text{ for } t \in (0,T)\,; \tag{2.24}$$

$$\text{with} \quad v(0) = v_0 \in H \quad \text{(the initial condition).} \tag{2.25}$$

Although in our parabolic evolutionary problems we make use of the monograph by A. PAZY [40], we would like to point to a closely related (and much newer) monograph by I. I. VRABIE [45] for interesting alternatives to results in [40].

In typical applications of our results to Mathematical Finance, in particular, to counterparty risk models treated in our present work, both, the coefficients in the *Black-Scholes operator* $\mathcal{A}(\tau)$ defined in eq. (2.3) and the *nonlinearity* $\tilde{F}(\,\cdot\,; x,\tau) : \mathbb{R} \to \mathbb{R}$ defined in eq. (2.4), are assumed to be continuous, or even Hölder-continuous, in time $\tau \in [0,T]$. In contrast, less restrictive time regularity hypotheses are needed in *Stochastic Control Theory*, such as piecewise continuity in time $\tau \in [0,T]$. Because our present work does not treat problems in "Stochastic Control Theory", we refer the interested reader to the monograph by A. BENSOUSSAN and J.-L. LIONS [8] for this interesting topic and for methods how to relax our time regularity hypotheses.

3 Monotone methods for the nonlinear B-S equation

We make use of the inhomogeneous linear problem (2.20), (2.2) in order to describe an iterative scheme for approximating the unique weak solution $v : [0,T] \to H$, $v(0) = v_0 \in H$, to the semilinear problem (2.13), (2.2).

3.1 Preliminary comparison results for parabolic problems

First, the so-called ***weak maximum principle*** for a *classical solution* v of the inhomogeneous linear problem (2.20), (2.2) is established e.g. in A. FRIEDMAN [22, Chapt. 2, Sect. 4, Theorem 9, p. 43]. A standard approximation procedure of a *weak solution* v by a sequence of classical solutions yields the corresponding *weak maximum principle* also for the *weak solution* v. More precisely, if the inequalities $v(x,0) = v_0(x) \geq 0$ and $f(x,t) \geq 0$ are valid for almost all $(x,t) \in \mathbb{R}^1 \times (0,T)$, then also $v(x,t) \geq 0$ holds for almost all $(x,t) \in \mathbb{R}^1 \times (0,T)$.

Second, let $v : [0,T] \to H$ be a *classical solution* of the inhomogeneous linear problem

$$\frac{\partial v}{\partial t} - \mathcal{A}(t)v + r_G\, v = g(t) \qquad \text{in } H \text{ for } t \in (0,T)\,; \tag{3.1}$$

$$\text{with} \quad v(0) = v_g \in H \quad \text{(the initial condition),} \tag{3.2}$$

where $g : [0,T] \to H$ is a function continuous and bounded in $(0,T)$, i.e., $g \in C((0,T) \to H)$ with $\|g\|_{L^\infty((0,T)\to H)} \stackrel{\text{def}}{=} \sup_{t\in(0,T)} \|g(t)\|_H < \infty$. We say that v is a ***supersolution*** to the inhomogeneous linear problem (2.20), (2.2), if the inequalities $v_g(x) \geq v_0(x)$ and $g(x,t) \geq f(x,t)$

hold for almost all $(x,t) \in \mathbb{R}^1 \times (0,T)$. Analogously, a ***subsolution*** v to the inhomogeneous linear problem (2.20), (2.2) is a *classical solution* v of problem (3.1), (3.2) for which the inequalities $v_g(x) \le v_0(x)$ and $g(x,t) \le f(x,t)$ hold for almost all $(x,t) \in \mathbb{R}^1 \times (0,T)$. More generally, if $v : (x,t) \in \mathbb{R}^1 \times (0,T) \to \mathbb{R}$ is a *weak solution* to the inhomogeneous linear problem (2.20), (2.2), then the notions of ***super-*** and ***subsolution*** to the inhomogeneous linear problem (2.20), (2.2) are defined by means of an approximation procedure by a sequence of classical solutions again. A rigorous functional-analytic way to define ***super-*** and ***subsolution*** v to problem (2.20), (2.2) is to require that $v : (x,t) \in \mathbb{R}^1 \times (0,T) \to \mathbb{R}$ have all regularity properties of a *weak solution* to problem (2.20), (2.2) stated in L. C. EVANS [19], Chapt. 7, §1.1, p. 352, or J.-L. LIONS [35], Chapt. IV, §1, p. 44, or [36], Chapt. III, eq. (1.11), p. 102, and, in addition to these regularity properties, the following inequality is valid in the *sense of distributions* on $\mathbb{R}^1 \times (0,T)$:

$$\frac{\partial v}{\partial t} - \mathcal{A}(t)v + r_G\, v \ge f(t) \quad (\le f(t)) \qquad \text{in } H \text{ for } t \in (0,T)\,; \tag{3.3}$$

$$\text{with} \quad v(0) \ge v_0 \in H \quad (\le v_0 \in H) \quad \text{(the initial condition).} \tag{3.4}$$

Here, the inequalities with "$\ge$" ("$\le$", respectively) specify a (weak) ***supersolution*** (a (weak) ***subsolution***). The reader is referred to A. FRIEDMAN [21, Chapt. 3, Sect. 3, Theorem 8, p. 51] for details about positive (or nonnegative) distributions. Clearly, any function $v : (x,t) \in \mathbb{R}^1 \times (0,T) \to \mathbb{R}$ which is simultaneously a (weak) ***supersolution*** and a (weak) ***subsolution***) of the inhomogeneous linear problem (2.20), (2.2) is a (weak) solution to this problem. Combining these definitions of ***super-*** and ***subsolution***, denoted by $\overline{v}, \underline{v} : \mathbb{R}^1 \times (0,T) \to \mathbb{R}$, respectively, having the initial values satisfying $\underline{v}(0) \le v_0 \le \overline{v}(0)$ a.e. in $\mathbb{R}^1$, with the weak maximum principle for the difference $w = \overline{v} - \underline{v}$, we obtain the following auxiliary ***weak comparison*** result.

Lemma 3.1 (Weak comparison.) *Assume that $\overline{v}, \underline{v} : \mathbb{R}^1 \times (0,T) \to \mathbb{R}$, respectively, is a pair of (weak)* ***super-*** *and* ***subsolutions*** *of problem (2.20), (2.2) satisfying $\underline{v}(0) \le \overline{v}(0)$ a.e. in $\mathbb{R}^1$. Then, at every time $t \in [0,T)$, we have $\underline{v}(x,t) \le \overline{v}(x,t)$ for a.e. $x \in \mathbb{R}^1$.*

Observe that we have left the initial value $v_0 \in H$ out of this lemma since we use it usually with either $\underline{v}(0) = v_0$ or $\overline{v}(0) = v_0$ as the initial condition attached to the differential equation

$$\frac{\partial \underline{v}}{\partial t} - \mathcal{A}(t)\underline{v} + r_G\, \underline{v} = \underline{f}(x,t) \quad \text{for } (x,t) \in \mathbb{R}^1 \times (0,T)\,, \tag{3.5}$$

or

$$\frac{\partial \overline{v}}{\partial t} - \mathcal{A}(t)\overline{v} + r_G\, \overline{v} = \overline{f}(x,t) \quad \text{for } (x,t) \in \mathbb{R}^1 \times (0,T)\,, \tag{3.6}$$

respectively, where $\underline{f}(\,\cdot\,,t) \le \overline{f}(\,\cdot\,,t)$ a.e. in $\mathbb{R}^1$, at every time $t \in (0,T)$.

Proof of **Lemma 3.1.** We subtract equation (3.5) from (3.6), thus obtaining an analogous equation for the difference $w = \overline{v} - \underline{v}$ with the right-hand side equal to $g(x,t) = \overline{f}(x,t) - \underline{f}(x,t) \ge 0$ for a.e. $(x,t) \in \mathbb{R}^1 \times (0,T)$. Then the desired result, $w(\,\cdot\,,t) \ge 0$ a.e. in $\mathbb{R}^1$, at every time

$t \in [0,T)$, follows from A. FRIEDMAN [22, Chapt. 2, Sect. 4, Theorem 9, p. 43], cf. also [22, Chapt. 2, Sect. 6, Theorem 16, p. 52]. ■

We now give simple **examples** of super- and subsolutions of problem (2.13), (2.2).

Example 3.2 (*Super-* and *subsolutions*.) Let us define the function

$$V(x,t) = K\,\mathrm{e}^{\lambda t}\left(\mathrm{e}^{\kappa x} + \mathrm{e}^{-\kappa x}\right) = 2K\,\mathrm{e}^{\lambda t}\cdot\cosh(\kappa x) \quad \text{for } (x,t)\in\mathbb{R}^1\times[0,T]\,, \tag{3.7}$$

where $\kappa \in \mathbb{R}$ is a constant satisfying $1 \le \kappa < \mu/2$, and $K, \lambda \in \mathbb{R}$ with $K \ge 1$ and $\lambda \ge 0$ are some other constants (large enough) to be determined below:

The left-hand side of eq. (2.13) with $v = V$ becomes

$$\begin{aligned}
&\text{l.h.s.}(x,t) = \frac{\partial V}{\partial t} - \mathcal{A}(t)V + r_G\,V(x,t)\\
&= \lambda\,V(x,t) - \frac{1}{2}\,\kappa^2[\sigma(t)]^2\,V(x,t)\\
&- \kappa\left[q_S(t) - \gamma_S(t) - \frac{1}{2}\,[\sigma(t)]^2\right]\cdot\frac{\mathrm{e}^{\kappa x}-\mathrm{e}^{-\kappa x}}{\mathrm{e}^{\kappa x}+\mathrm{e}^{-\kappa x}}\,V(x,t) + r_G\,V(x,t)\\
&\ge \left[(\lambda + r_G) - \frac{1}{2}\,\kappa^2[\sigma(t)]^2\right]V(x,t) - \kappa\cdot\left|q_S(t) - \gamma_S(t) - \frac{1}{2}\,[\sigma(t)]^2\right|\,V(x,t)\\
&\ge \left\{(\lambda + r_G) - \tfrac{1}{2}\,\kappa^2\|\sigma\|^2_{L^\infty(0,T)} - \kappa\left[\|q_S - \gamma_S\|_{L^\infty(0,T)} + \tfrac{1}{2}\,\|\sigma\|^2_{L^\infty(0,T)}\right]\right\}V(x,t)\\
&\quad \text{for } (x,t)\in\mathbb{R}^1\times[0,T]\,.
\end{aligned} \tag{3.8}$$

As usual, $\|\sigma\|_{L^\infty(0,T)}$ stands for the supremum norm of a continuous function $\sigma : [0,T] \to \mathbb{R}$.

On the other hand, the right-hand side of eq. (2.13) with $v = V$ becomes

$$\begin{aligned}
&\text{r.h.s.}(x,t) = G(V(x,t);x,t) = G(0;x,t) + [G(V(x,t);x,t) - G(0;x,t)]\\
&\le C_0\,\exp(|x|) + L_G\,V(x,t) \le C_0'\,V(x,t) \quad \text{for } (x,t)\in\mathbb{R}^1\times[0,T]\,,\\
&\quad \text{where} \quad C_0' \stackrel{\text{def}}{=} (C_0/K) + L_G \quad (\,>0)\,,
\end{aligned} \tag{3.9}$$

and we have taken advantage of inequalities (2.10) and (2.11) in Hypotheses (**G2**) and (**G4**), respectively. Subtracting eq. (3.9) from (3.8) we arrive at

$$\begin{aligned}
&\text{l.h.s.}(x,t) - \text{r.h.s.}(x,t) = \frac{\partial V}{\partial t} - \mathcal{A}(t)V + r_G\,V(x,t) - G(V(x,t);x,t)\\
&\ge \Big\{\lambda + r_G - \tfrac{1}{2}\,\kappa^2\|\sigma\|^2_{L^\infty(0,T)}\\
&\qquad - \kappa\left[\|q_S - \gamma_S\|_{L^\infty(0,T)} + \tfrac{1}{2}\,\|\sigma\|^2_{L^\infty(0,T)}\right] - C_0'\Big\}V(x,t) \ge 0\\
&\quad \text{for } (x,t)\in\mathbb{R}^1\times[0,T]\,,
\end{aligned} \tag{3.10}$$

provided $\lambda \in \mathbb{R}$ satisfies

$$\begin{aligned}
\lambda + r_G \ge \Lambda &\stackrel{\text{def}}{=} \tfrac{1}{2}\,\kappa^2\|\sigma\|^2_{L^\infty(0,T)}\\
&\quad + \kappa\left[\|q_S - \gamma_S\|_{L^\infty(0,T)} + \tfrac{1}{2}\,\|\sigma\|^2_{L^\infty(0,T)}\right] + C_0' \quad (\,>0)\,.
\end{aligned} \tag{3.11}$$

Recalling our Hypothesis ($\mathbf{v_0}$) with ineq. (2.17) on the growth of the initial condition, we take the constant $K \in \mathbb{R}$ such that $K \geq \max\{1,\, C_h\}$ which guarantees also

$$\begin{aligned} |v_0(x)| &= |h(\mathrm{e}^x)| \leq C_h \cdot \exp(|x|) \leq C_h \cdot \exp(\kappa |x|) \\ &\leq V(x,0) = K\left(\mathrm{e}^{\kappa x} + \mathrm{e}^{-\kappa x}\right) = 2K \cdot \cosh(\kappa x) \quad \text{for } (x,t) \in \mathbb{R}^1 \times [0,T]\,. \end{aligned} \tag{3.12}$$

It follows that the function $V : \mathbb{R}^1 \times [0,T] \to \mathbb{R}$ defined in eq. (3.7) is a *supersolution* of problem (2.13), (2.2).

Analogous arguments show that the function $-V : \mathbb{R}^1 \times [0,T] \to \mathbb{R}$ is a *subsolution* of problem (2.13), (2.2). Notice that in this case, ineq. (3.9) has to be replaced by

$$\begin{aligned} & G(-V(x,t);x,t) = G(0;x,t) + [G(-V(x,t);x,t) - G(0;x,t)] \\ & \geq -C_0'\, V(x,t) \quad \text{for } (x,t) \in \mathbb{R}^1 \times [0,T]\,. \end{aligned} \tag{3.13}$$

□

3.2 Construction of monotone iterations

Recalling Lemma 3.1, our Hypothesis ($\mathbf{v_0}$) with ineq. (2.17), and applying Example 3.2 with $\kappa = 1$ $(< \mu/2)$, we are now ready to construct a ***monotone iteration scheme*** for calculating a (weak) solution $v : \mathbb{R}^1 \times (0,T) \to \mathbb{R}$ to the initial value problem (2.13), (2.2). We start by setting $\kappa = 1$ and fixing the constants $K, \lambda \in \mathbb{R}$ with $K \geq 1$ and $\lambda + r_G \geq \Lambda$ (> 0) large enough, such that both inequalities, (3.11) and (3.12), are valid. It follows from eq. (3.7) and ineq. (3.10) that the function $u_0 : \mathbb{R}^1 \times (0,T) \to \mathbb{R}$ defined by

$$u_0(x,t) = K\,\mathrm{e}^{\lambda t}\left(\mathrm{e}^{x} + \mathrm{e}^{-x}\right) = 2K\,\mathrm{e}^{\lambda t} \cdot \cosh(x) \quad \text{for } (x,t) \in \mathbb{R}^1 \times [0,T]\,, \tag{3.14}$$

is a *supersolution* of problem (2.13), (2.2). We remark that $-u_0$ happens to be a *subsolution* of this problem, by Example 3.2 with $\kappa = 1$, as well. The first iterate, $u_1 : \mathbb{R}^1 \times [0,T] \to \mathbb{R}$, is constructed as the (weak) solution u_1 to the following analogue of the inhomogeneous linear initial value problem (2.20), (2.2):

$$\begin{aligned} \frac{\partial u_1}{\partial t} - \mathcal{A}(t) u_1 + r_G\, u_1 &= G(u_0(x,t);x,t) \\ & \left(\leq C_0'\, u_0(x,t) \right) \qquad \text{for } (x,t) \in \mathbb{R}^1 \times [0,T]\,; \end{aligned} \tag{3.15}$$

$$u_1(x,0) = v_0(x) \stackrel{\text{def}}{=} h(\mathrm{e}^x) \qquad \text{for } x \in \mathbb{R}^1\,. \tag{3.16}$$

Since u_0 is a (weak) supersolution of problem (2.13), (2.2), by Example 3.2 with $\kappa = 1$, we may apply Lemma 3.1 to conclude that $u_1(x,t) \leq u_0(x,t)$ holds for a.e. $(x,t) \in \mathbb{R}^1 \times (0,T)$. In addition, making use of eq. (3.13), we get also

$$\begin{aligned} -u_0(x,t) \leq u_1(x,t) \leq u_0(x,t) \quad & \left(= K\,\mathrm{e}^{\lambda t}\left(\mathrm{e}^{\kappa x} + \mathrm{e}^{-\kappa x}\right)\right) \\ & \text{for } (x,t) \in \mathbb{R}^1 \times [0,T]\,. \end{aligned} \tag{3.17}$$

Our next step is the following *induction hypothesis*. Let us assume that, for some integer $m \geq 1$, in addition to u_0 and u_1 above, we have already constructed the first $(m+1)$ functions $u_0, u_1, u_2, \dots, u_m : \mathbb{R}^1 \times [0,T] \to \mathbb{R}$ with the following properties:

(a) Every function $u_j : \mathbb{R}^1 \times [0,T] \to \mathbb{R}$; $j = 0,1,2,\dots,m$, is Lebesgue-measurable and continuous in time as a function $u_j : [0,T] \to H$, i.e., $u_j \in C([0,T] \to H)$.

(b) The inequalities

$$-u_0(x,t) \le u_j(x,t) \le u_{j-1}(x,t) \le u_0(x,t) \quad \text{for } (x,t) \in \mathbb{R}^1 \times [0,T]\,, \tag{3.18}$$

are valid for every $j = 1,2,3,\dots,m$.

(c) For each $j = 1,2,3,\dots,m$, the function $u_j : \mathbb{R}^1 \times [0,T] \to \mathbb{R}$ is the (weak) solution to the following analogue of the inhomogeneous linear initial value problem (2.20), (2.2); cf. problem (3.15), (3.16) above:

$$\frac{\partial u_j}{\partial t} - \mathcal{A}(t)u_j + r_G\, u_j = G(u_{j-1}(x,t);x,t) \tag{3.19}$$
$$\big(\le C_0'\, u_0(x,t)\big) \qquad \text{for } (x,t) \in \mathbb{R}^1 \times [0,T]\,;$$

$$u_j(x,0) = v_0(x) \stackrel{\text{def}}{=} h(\mathrm{e}^x) \qquad \text{for } x \in \mathbb{R}^1\,. \tag{3.20}$$

In our last step (*induction on the index* $m \ge 1$) we construct the $(m+1)$-st iterate, $u_{m+1} : \mathbb{R}^1 \times [0,T] \to \mathbb{R}$, to be the (weak) solution u_{m+1} to the following problem; cf. problem (3.15), (3.16):

$$\frac{\partial u_{m+1}}{\partial t} - \mathcal{A}(t)u_{m+1} + r_G\, u_{m+1} = G(u_m(x,t);x,t) \tag{3.21}$$
$$\big(\le C_0'\, u_0(x,t)\big) \qquad \text{for } (x,t) \in \mathbb{R}^1 \times [0,T]\,;$$

$$u_{m+1}(x,0) = v_0(x) \stackrel{\text{def}}{=} h(\mathrm{e}^x) \qquad \text{for } x \in \mathbb{R}^1\,. \tag{3.22}$$

By arguments analogous to those used in the construction of u_1 from u_0 above, we conclude that u_{m+1} exists and satisfies

$$-u_0(x,t) \le u_{m+1}(x,t) \le u_m(x,t) \le u_0(x,t) \quad \text{for } (x,t) \in \mathbb{R}^1 \times [0,T]\,. \tag{3.23}$$

Here, we have used our monotonicity hypothesis (**G3**) to conclude that $u_m \le u_{m-1}$ a.e. in $\mathbb{R}^1 \times [0,T]$ entails $G(u_m(x,t);x,t) \le G(u_{m-1}(x,t);x,t)$ for a.e. $(x,t) \in \mathbb{R}^1 \times [0,T]$. Finally, we get $u_{m+1} \in C([0,T] \to H)$. This concludes the construction of the desired iterates.

Remark 3.3 (Inhomogeneous linear problem.) An ***explicit formula*** for calculating the weak solution $u_{m+1} : \mathbb{R}^1 \times [0,T] \to \mathbb{R}$ to problem (3.21), (3.22) will be described later in Section 5, Corollary 5.2. Numerical methods for computing this solution will be discussed in Remark 5.4, as well. □

As an obvious consequence of our construction we conclude that, in addition to u_0, also each iterate u_j; $j = 1,2,3,\dots$, is a (weak) supersolution of problem (2.13), (2.2).

Standard application of Lebesgue's monotone (or dominated) convergence theorem yields the monotone pointwise convergence $u_m(x,t) \searrow v(x,t)$ as $m \nearrow \infty$ for a.e. $(x,t) \in \mathbb{R}^1 \times [0,T]$, as well as the L^2-type norm convergence $\|u_m(t) - v(t)\|_H \searrow 0$ as $m \nearrow \infty$, for a.e. $t \in (0,T)$, where $v(t) \equiv v(\,\cdot\,,t) \in H$ satisfies

$$-\,u_0(x,t) \leq v(x,t) \leq u_m(x,t) \leq u_0(x,t) \quad \text{for } (x,t) \in \mathbb{R}^1 \times [0,T]\,. \tag{3.24}$$

Furthermore, we get another L^2-type norm convergence in the Lebesgue(-Hilbert) space $L^2([0,T] \to H)$, namely,

$$\|u_m - v\|_{L^2([0,T]\to H)} \stackrel{\text{def}}{=} \left(\int_0^T \|u_m(t) - v(t)\|_H^2 \,\mathrm{d}t \right)^{1/2} \searrow 0 \quad \text{as } m \nearrow \infty\,. \tag{3.25}$$

We combine this result with Theorem 1.2 (and its proof) in A. PAZY [40, Chapt. 6, §6.1, pp. 184–185] to conclude that $v \in C([0,T] \to H)$ and $v : \mathbb{R}^1 \times [0,T] \to \mathbb{R}$ is a ***mild solution*** to the initial value problem (2.13), (2.2). Finally, applying the well-known results from L. C. EVANS [19], Chapt. 7, §1.1, p. 352, or J.-L. LIONS [35], Chapt. IV, §1, p. 44, or [36], Chapt. III, eq. (1.11), p. 102, we find out that v is a (weak) solution to problem (2.13), (2.2).

We summarize the most important results from our monotone iteration scheme (3.14) – (3.23) for problem (2.13), (2.2) in the following theorem. Precise definitions of the Hölder spaces used in this theorem, $H^{\theta,\theta/2}\big(D_{1+1}^{(T)}\big)$ (a local Hölder space) and $H^{2+\theta,1+(\theta/2)}(\overline{Q'})$ over the parabolic domain $D_{1+1}^{(T)} = \mathbb{R}^1 \times (0,T)$ (an open strip in $\mathbb{R}^1 \times \mathbb{R}$) and its compact subset $\overline{Q'} = [a,b] \times [\tau, T'] \subset D_{1+1}^{(T)}$ (a compact rectangle), respectively, with $\theta \in (0,1)$, $-\infty < a < b < +\infty$, and $0 < \tau < T' < T$, can be found in O. A. LADYZHENSKAYA, V. A. SOLONNIKOV, and N. N. URAL'TSEVA [32, Chapt. I, §1, pp. 7–8].

Theorem 3.4 (Monotone iterations.) *Let $v_0 \in H$ obey* Hypothesis ($\mathbf{v_0}$) *with ineq.* (2.17)*. Then the monotone iterations $u_0 \geq u_1 \geq \cdots \geq u_{j-1} \geq u_j \geq \cdots \geq -\,u_0$, described in items* (a)*,* (b)*, and* (c) *above, converge in the Lebesgue(-Hilbert) space $L^2([0,T] \to H)$ to a function $v : \mathbb{R}^1 \times (0,T)$ according to formula* (3.25)*. The limit function, $v \in L^2([0,T] \to H)$, is a (weak) solution to problem* (2.13)*,* (2.2)*. Furthermore, there is a constant $\theta \in (0,1)$ such that $u_m \in H^{\theta,\theta/2}\big(D_{1+1}^{(T)}\big)$ holds for every $m = 1,2,3,\dots,$ and $v \in H^{\theta,\theta/2}\big(D_{1+1}^{(T)}\big)$, as well.*

Finally, assume that the function

$$\begin{aligned} &G(v;x,\,\cdot\,) : t \mapsto G(v;x,t) : \mathbb{R}^1 \times (0,T) \to \mathbb{R} \\ &\quad \textit{is uniformly Hölder-continuous on } (0,T), \\ &\quad \textit{uniformly for } (v,x) \textit{ in every bounded subset of } \mathbb{R} \times \mathbb{R}^1\,. \end{aligned} \tag{G1'}$$

Then we get even $u_m \in H^{2+\theta,1+(\theta/2)}\big(D_{1+1}^{(T)}\big)$ for every $m = 1,2,3,\dots,$ together with $v \in H^{2+\theta,1+(\theta/2)}\big(D_{1+1}^{(T)}\big)$, where the convergence $u_m \to v$ holds in the norm of the Hölder space $H^{2+\theta',1+(\theta'/2)}(\overline{Q'})$ over any compact rectangle $\overline{Q'} = [a,b] \times [\tau,T'] \subset D_{1+1}^{(T)}$ in the open strip $D_{1+1}^{(T)} = \mathbb{R}^1 \times (0,T)$, with $-\infty < a < b < +\infty$ and $0 < \tau < T' < T$, and with any Hölder exponent $\theta' \in (0,\theta)$. In particular, each function u_m ($m = 1,2,3,\dots$) is a strong (classical) solution *to problem* (3.19)*,* (3.20)*, whereas v is a* strong (classical) solution *to problem* (2.13)*,* (2.2)*.*

Remark 5.4 **(i)** Eq. (2.20) being a linear *"diffusion equation"*, the Monte Carlo method is a very natural way for approximating the precise analytic solution to problem (2.20), (2.2) (i.e., (2.24), (2.25)) by numerical simulations produced by this method. We refer to the works by I. ARREGUI, B. SALVADOR, and C. VÁZQUEZ [3, Sects. 3–4, pp. 18–23], F. BAUSTIAN, M. FENCL, J. POSPÍŠIL, and V. ŠVÍGLER [6, §3.6, p. 52] and to M. YU. PLOTNIKOV [41, Sect. 1, pp. 121–125] for greater details. The first two references, [3, 6], treat exactly the problem of *"Option pricing under some Value Adjustment"* (***xVA***) in Mathematical Finance; under *"Credit Value Adjustment"* (***CVA***), for instance. The third one, [41], treats linear integral equations of type (5.16) which envolve the evolutionary family of bounded linear integral operators $\mathfrak{T}(t,s) : H \to H$ $(0 \leq s < t \leq T)$ on the Hilbert space H. In fact, a numerical method for solving the full, nonlinear integral equation (5.1) for the unknown function $v \in C([0,T] \to H)$ substituted into the (subsequently unknown) nonlinearity

$$f(\tau) \equiv f(\,\cdot\,,\tau) : \mathbb{R}^1 \to \mathbb{R} : x \mapsto f(x,\tau) = G(v(x,\tau);x,\tau) \quad \text{with } f(\tau) \in H$$

for every $\tau \in [0,T]$, is provided in [41]. This work is based in an iteration method for a nonlinear integral equation similar to ours, cf. [41, Eq. (1.1), p. 121].

(ii) In contrast with the probabilistic Monte Carlo methods for solving the Cauchy problem (2.20), (2.2), analytic methods based on a ***finite difference (or finite element) scheme*** provide a highly competitive alternative to Monte Carlo in a series of works, such as the monograph by Y. ACHDOU and O. PIRONNEAU [1] and the articles by I. ARREGUI et AL. [3, §3.4, pp. 20–21], [4, Sect. 4, pp. 734–737], F. BAUSTIAN et AL. [6, §3.5, pp. 50–51], M. N. KOLEVA [30, Sect. 3, pp. 367–368], and M. N. KOLEVA and L. G. VULKOV [31, Sects. 3–4, pp. 510–515].

(iii) Last but not least, a "hybrid" algorithm mixing ***Monte Carlo*** with ***finite difference/element*** methods in quest for optimization on both, precision and speed, is presented in G. LOEPER and O. PIRONNEAU [37]. □

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