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Semi-linear and quasi-linear Black-Scholes-type equations in Mathematical Finance: Analytical and numerical methods with monotonicity

Lecture 2:

Option pricing with transaction costs and a fully nonlinear Black-Scholes equation

Peter Takáč
Universität Rostock
D-18055 Rostock, Germany

Joint work with **Bénédicte Alziary** (T.S.E.)

These lectures will be posted on the website
(with corrections and hopefully also improvements):

peter.takac@uni-rostock.de

https://www.mathematik.uni-rostock.de/
unser-institut/professuren-apl-prof/
angewandte-analysis/invited-lectures/

In a market with transaction costs, generally, there is no nontrivial portfolio that dominates a contingent claim. Therefore, in such a market, preferences have to be introduced in order to evaluate the prices of options. We will discuss the model in G. Barles and H. M. Soner (1998) which quantifies this dependence on preferences in the specific example of a European call option. This is achieved by using the utility function approach of Hodges and Neuberger together with an asymptotic analysis of partial differential equations. The result is

a nonlinear Black-Scholes equation with an adjusted volatility which is a function of the second derivative of the price itself. In this model, our **attitude towards risk** is summarized in one free parameter, $\boldsymbol{a}$,
which appears in the **non-linear** Black-Scholes-equation.
A comprehensive derivation is in the article by
Guy Barles and **Halil Mete Soner**:
Option pricing with transaction costs and
a nonlinear Black-Scholes equation,
Finance and Stochastics, **2** (1998), pp. 369-397.

In a market with transaction costs, generally, there is no nontrivial portfolio that dominates a contingent claim. Therefore, in such a market, preferences play an important role.

Option pricing with transaction costs introduced in **H. E. Leland** (1985) asumes the convexity of the resulting option price. Technically, it makes use of short time increments of the standard one-dimensional Brownian motion

$$W(t+\Delta t) - W(t) \approx \sqrt{\frac{2}{\pi}}\sqrt{\Delta t} \approx c^*\sqrt{\Delta t}$$

with an arbitrary constant c^*. The "optimal" choice of c^* is an interesting question related to the **risk** inherent in markets with transaction costs.

H. E. Leland: *Option pricing and replication with transaction costs*, Journal of Finance, **40** (1985), pp. 1283-1301.

$\Longrightarrow$ An adjustment to the constant volatility $\sigma > 0$,

$$\partial(c^*) = \sigma\left(1 + c^*\frac{\mu}{\sigma\sqrt{\Delta t}}\right)^{1/2}$$

(implied volatility ?) with an arbitrary constant c^*.

In a market with transaction costs, generally, there is no nontrivial portfolio that dominates a contingent claim.

S. D. Hodges and A. Neuberger: *Optimal replication of contingent claims under transaction costs*, Review of Futures Markets, **8** (1989), pp. 222-239.

Hodges & **Neuberger** consider the difference between the maximum utility from final wealth
(i) when there is no option liability and
(ii) when there is such a liability.

They postulate: The price of the option $=$ the unique cash increment which offsets this difference.

A **consequence** in the absence of market frictions:
Option price obtained from utility maximization
$=$ the Black-Scholes option price.

Barles & Soner use the **exponential utility function**

$$U^{\epsilon}(\xi) \stackrel{\text{def}}{=} 1 - \exp\left(-\frac{\xi}{\epsilon}\right) \qquad \text{for } \xi \in \mathbb{R}_1\,, \quad \text{where } \epsilon > 0\,,$$

where $1/\epsilon =$ (the product of)
risk-aversion factor $\times$ number of options to be sold.

Notation: $\mu > 0$ − proportional transaction cost
$S > 0$ − Stock price at time $t \leq T$
$\Psi^{\epsilon,\mu}(S,t)$ − Option price with utility function U^{ϵ}
$\epsilon \to 0+$ and $\mu \to 0+$ under the restriction

$$\frac{\mu}{\sqrt{\epsilon}} = a > 0 \qquad \text{is a constant.}$$

Calculate the parabolic PDE for the limit
$\Psi^{\epsilon,\mu}(S,t) \longrightarrow \Psi(S,t)$ as $\mu = a\sqrt{\epsilon} \to 0+$:
The (fully) nonlinear Black-Scholes equation.

Notation: The **model** by Barles & Soner is defined on a filtered probability space $\left(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P}\right)$, where $\mathbb{P}$ is a risk neutral probability measure and the filtration $(\mathscr{F}_t)_{t\geq 0}$ satisfies the usual conditions.

- $W(\cdot)$ – a standard 1-dimensional Brownian motion

Financial market:

one money market & one stock market;

- $S(\cdot)$ – price of stock satisfies the SDE

$$\text{(1)} \qquad \begin{array}{l} \mathrm{d}S(\tau) = S(\tau)\left[\alpha\,\mathrm{d}\tau + \sigma\,\mathrm{d}W(\tau)\right], \quad \tau \in [t,T]\,; \\ \text{with the initial data } S(t) = s \in \mathbb{R}\,. \end{array}$$

Two processes:

- $X(\cdot)$ – \$\$\$ holdings (money market),
- $Y(\cdot)$ – shares of stock owned.

The initial values $X(t) = x \in \mathbb{R}$ and $Y(t) = y \in \mathbb{R}$.

Trading strategy for $X(\cdot)$ and $Y(\cdot)$:

- $L(\cdot)$ – transfers money $\longrightarrow$ stock,
- $M(\cdot)$ – transfers $\longleftarrow$ (in shares of stock).

The initial values $L(t) = M(t) = 0$.

Portfolio evolves:

$$X(\tau) = X\left(\tau; t, x, y, L(\cdot), M(\cdot)\right),$$
$$Y(\tau) = Y\left(\tau; t, x, y, L(\cdot), M(\cdot)\right).$$

- Proportional transaction cost $\mu \in (0, 1)$,

$X(t) = x$, $Y(t) = y$.

$$X(\tau) = x - \int_t^\tau S(\theta)\, \mathrm{d}L(\theta) + \int_t^\tau S(\theta)\, \mathrm{d}M(\theta) - \mu \left\{ \int_t^\tau S(\theta)\, \mathrm{d}L(\theta) + \int_t^\tau S(\theta)\, \mathrm{d}M(\theta) \right\}$$

(2)
$$X(\tau) = x - \int_t^\tau S(\theta) \cdot (1 + \mu)\, \mathrm{d}L(\theta) + \int_t^\tau S(\theta) \cdot (1 - \mu)\, \mathrm{d}M(\theta), \quad \tau \in [t, T],$$

(3)
$$Y(\tau) = y + L(\tau) - M(\tau), \quad \tau \in [t, T].$$

• $U : \mathbb{R} \to \mathbb{R}$ − a **utility function**,
i.e., nondecreasing and concave.

• **Value function**:
if there are NO option liabilities, then it is

$$V^{\text{free}}(x, y, S, t) \equiv V^{\text{f}}(x, y, S, t) \stackrel{\text{def}}{=} \sup_{L(\,\cdot\,), M(\,\cdot\,)} \mathsf{E}\left\{U\left(X(T) + Y(T)\,S(T)\right)\right\} . \tag{4}$$

if N ($N \geq 1$) European call options have been sold, then it is

$$V(x, y, S, t) \stackrel{\text{def}}{=} \sup_{L(\,\cdot\,), M(\,\cdot\,)} \mathsf{E}\left\{U\left(X(T) + Y(T)\,S(T) - N\,(S(T) - K)^{+}\right)\right\} , \tag{5}$$

since our **final wealth** at time $t = T$ equals to

$$U \quad \left(X(T) + Y(T)\,S(T) - N \cdot (S(T) - K)^{+} .\right)$$

Hodges & **Neuberger** postulate:
The price of each option is equal to
the **maximal solution** Λ of the *algebraic equation*
(in fact, typically the unique solution)

(6) $$V(x + N\Lambda, y, S, t) = V^{\mathsf{f}}(x, y, S, t)\,.$$

This solution, $\Lambda = \Lambda(x, y, S, t)$, depends NOT only on the initial data (x, y, S, t), but also on N and the utility function $U(\,\cdot\,)$.

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The **remaining part** of the lecture is concerned with rather **detailed explanations** of the arguments used in the article

Guy Barles and **Halil Mete Soner**:
Option pricing with transaction costs and
a nonlinear Black-Scholes equation,
Finance and Stochastics, **2** (1998), pp. 369-397.

In particular, those used on pages 372 − 379.