Time-fractional stochastic conservation laws

(joint work with Martin Scholtes)

We consider time-fractional stochastic scalar conservation laws of the form

$$dg_{1-\alpha} * (u - u_0) + \text{ div } f(u)dt = I^{1-\beta} hdW \qquad (*)$$

where $g_{1-\alpha}(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$ $(\alpha \in (0,1))$, i.e.,

$$\partial_t g_{1-\alpha} * (u - u_0) = \partial_t^\alpha (u - u_0)$$

is the fractional time-derivative in the sense of Riemann-Liouville,

$$f : \mathbb{R} \to \mathbb{R}^N \text{ is a smooth function, and}$$

$I^{1-\beta}$ is the fractional integral of order $1-\beta$ in the sense of Riemann-Liouville ($\beta = 1$ corresponds to the classical additive stochastic noise hdW with a given function h and $W = (W(t), \mathcal{F}_t, 0 \le t \le T)$ one-dimensional Brownian motion on a classical Wiener space).

Under certain assumptions on α and β we prove existence and uniqueness of stochastic entropy solutions for arbitrary L^2-initial data.

An interesting open question is whether it is possible to generalize these results to the case of a multiplicative stochastic noise. The main difficulty is that an Itô type formula is not known to exist in the time-fractional derivative case.

In this talk, before studying the time-fractional stochastic conservation law $(*)$, we will give a short introduction into the theory of fractional derivatives (which dates back to Leibniz who, already in 1695, proposed how to define the fractional derivative $\frac{d^{1/2}}{dt^{1/2}}$ of a function). In the following we will also give an elementary introduction into stochastic perturbations (SDEs/SPDEs) and scalar conservation laws. With these tools at hand we go on and study problem $(*)$.