Lösungen von parabolischen Anfangs- und Randwertaufgaben mit dem p-Laplaceoperator ($1 < p < \infty$), welche einen kompakten Träger mit mehreren Zusammenhangskomponenten haben

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A parabolic problem with

degenerate diffusion and nonsmooth reaction

We look for nonnegative solutions $u : \Omega \times (0,T) \to \mathbb{R}_+$

$$(1)\quad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta_p u = q(x)\, u(x,t)^\alpha & \text{for } x \in \Omega\,,\ 0 < t < T\,; \\ \qquad u(x,t) = 0 & \text{for } x \in \partial\Omega\,,\ 0 < t < T\,. \end{cases}$$

- $2 < p < \infty$ and $0 < \alpha < 1$

J. Benedikt, P. Girg, L. Kotrla, and P. Takáč (2016)

First paper on the subject:

- $p = 2$ and $0 < \alpha < 1$

H. Fujita and S. Watanabe in C.P.A.M. (1968)

$\Delta_p u \stackrel{\text{def}}{=} \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ – the p-Laplace operator.

General Problem:

(a) Let us begin with a pair (u, v) of weak sub- and super-solutions
$u, v \in C\left([0, T] \to L^2(\Omega)\right) \cap L^p\left((0, T) \to W_0^{1,p}(\Omega)\right)$
with the time derivatives

$$\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t} \in L^{p'}\left((0, T) \to W^{-1,p'}(\Omega)\right), \qquad \frac{1}{p} + \frac{1}{p'} = 1,$$

such that u and v, respectively, satisfy the following inequalities in the sense of distributions in $L^{p'}\left((0, T) \to W^{-1,p'}(\Omega)\right)$:

$$\text{(2)} \qquad \frac{\partial u}{\partial t} - \Delta_p u \leq f(x, t, u(x, t)) \quad \text{in } \Omega \times (0, T),$$

$$\text{(3)} \qquad \frac{\partial v}{\partial t} - \Delta_p v \geq f(x, t, v(x, t)) \quad \text{in } \Omega \times (0, T);$$

$$\text{(4)} \qquad u(x, 0) \leq v(x, 0) \quad \text{in } \Omega.$$

The Weak Comparison Principle:

If $s \mapsto f(x,t,s) : \mathbb{R} \to \mathbb{R}$ satisfies
a **one-sided Lipschitz condition**,
$s \mapsto f(x,t,s) - Ls : \mathbb{R} \to \mathbb{R}$ monotone decreasing
for some constant $L \geq 0$, then the operator
$u \mapsto -\Delta_p u - f(x,t,u(x,t)) + Lu : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$
is monotone and the inequality $u \leq v$ in $\Omega \times (0,T)$
follows by a standard variational method with
the test function $(u-v)^+$.

$\Longrightarrow$ Uniqeness of Weak Solution.

Example. $s \mapsto s^\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ does not satisfy
a *one-sided Lipschitz condition* if $0 < \alpha < 1$.

$$(1)\quad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta_p u = q(x)\, u(x,t)^\alpha & \text{for } x \in \Omega\,,\ 0 < t < T\,; \\ u(x,t) = 0 & \text{for } x \in \partial\Omega\,,\ 0 < t < T\,. \end{cases}$$

The Strong Comparison Principle:
For instance, assume that $s \mapsto f(x,t,s) : \mathbb{R} \to \mathbb{R}$ satisfies
a **one-sided Lipschitz condition**,
so that the Weak Comparison Principle is valid,
$u \leq v$ in $\Omega \times (0,T)$.

Open QUESTION:

Under what conditions on $p \in (1,\infty)$ ($p \neq 2$),
the domain $\Omega \subset \mathbb{R}^N$, $f(x,t,s)$, the initial data
$u(x,0)$, $v(x,0)$, and the regularity of $u,v : \Omega \times (0,T) \to \mathbb{R}$
is the following (Hopf's) version of
the Strong Comparison Principle valid:

If (u, v) is a pair of weak sub- and super-solutions satisfying inequalities (2), (3), such that
$u(x,0) \leq v(x,0)$ for all $x \in \Omega$, $u(\,\cdot\,,0) \not\equiv v(\,\cdot\,,0)$,

then we have Hopf's Comparison Principle,

$$\begin{aligned} u(x,t) &< v(x,t) && \text{for all } (x,t) \in \Omega \times (0,T) \quad \text{and} \\ \frac{\partial u}{\partial \nu}(x,t) &> \frac{\partial v}{\partial \nu}(x,t) && \text{for all } (x,t) \in \partial\Omega \times (0,T)\,. \end{aligned}$$

Example. True if the linearization for $w = u - v$ is a regular parabolic equation that admits a classical solution $w \in C^{2,1}(\Omega \times (0,T)) \cap C^{1,0}(\overline{\Omega} \times (0,T))$, by Hopf's Maximum Principle for linear problems ($p = 2$).
Typically, this requires the "degeneracy / singularity" set
$\{(x,t) \in \Omega \times (0,T) : \nabla u(x,t) = \nabla v(x,t) = \mathbf{0}\}$
to be empty (hard to verify, in general).

(b) **Counterexamples** to both,
weak and strong comparison principles,
are frequently obtained from
the **weak and strong maximum principles**,
$u \equiv 0 \leq v(x,t)$ in $\Omega \times (0,T)$.

A few more papers (besides our J.D.E. (2016) paper):

- $2 < p < \infty$ and $0 < \alpha < 1$

A. de Pablo and J.-L. Vázquez (1990, 1992)
B. Nazaret in C.R.A.S. (2001)
E. DiBenedetto, U. Gianazza, and V. Vespri (2012)
V. E. Bobkov and P. Takáč (2014)
S. Merchán, L. Montoro, and I. Peral (2015)
V. E. Bobkov, J. Benedikt, P. Girg, L. Kotrla, and P. Takáč (2015)
V. E. Bobkov and P. Takáč (??)

- $p = 2$ and $0 < \alpha < 1$

H. Fujita and S. Watanabe in C.P.A.M. (1968).

The Strong Maximum Principle: $0 \le u(x,t)$ in eq. (1),

$$(1) \quad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta_p u = q(x)\, u(x,t)^{\alpha} & \text{for } x \in \Omega\,,\ 0 < t < T\,; \\ u(x,t) = 0 & \text{for } x \in \partial\Omega\,,\ 0 < t < T\,. \end{cases}$$

Theorem 1 ($N = 1$). Let $2 < p < \infty$, $1/(p-1) < \alpha < 1$, and let $I_k = [a_k, b_k]$; $k = 1, 2, 3, \dots, m$,
be a family of pairwise disjoint compact intervals in $\mathbb{R}$,
$-\infty < a_1 < b_1 < a_2 < b_2 < \cdots < a_m < b_m < +\infty$,
and let $0 < T_0 < \infty$. Furthermore, let
$\xi_k \in (a_k, b_k)$ be an arbitrary point; $k = 1, 2, 3, \dots, m$.

Then there exists some $T \in (0, T_0]$ such that the initial-boundary value problem

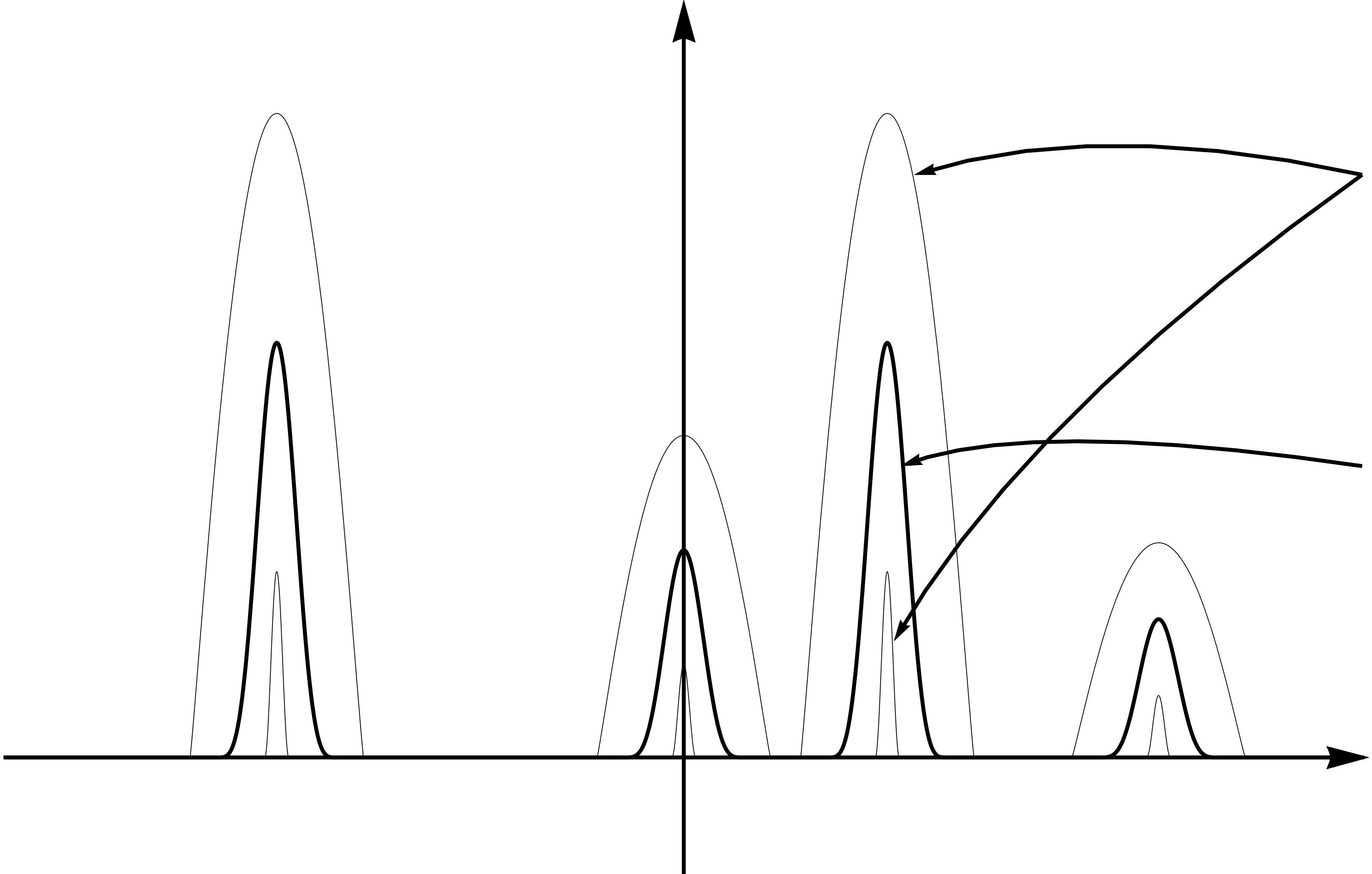
$$\text{(5)} \quad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta_p u = u(x,t)^\alpha \,, & x \in (a_1, b_m)\,,\ 0 < t < T\,; \\ u(a_1, t) = u(b_m, t) = 0\,, & 0 < t < T\,; \\ u(x, 0) = 0\,, & x \in (a_1, b_m)\,, \end{cases}$$

possesses a nontrivial nonnegative solution $u : (a_1, b_m) \times (0, T) \to \mathbb{R}_+$ such that

(i) $u(\xi_k, t) > 0$ for all $k = 1, 2, \ldots, m$ and all $t \in (0, T)$;

(ii) $u(x, t) = 0$ for all $x \in \mathbb{R} \setminus \cup_{k=1}^m (a_k, b_k)$ and all $t \in (0, T)$.

Hence, u is a multi-bump solution with at least m bumps.



Theorem 2 ($N \geq 2$). Let $p > 2$, $1/(p-1) < \alpha < 1$, and let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary. Assume that $q : \Omega \to \mathbb{R}$ satisfies hypothesis (Q),

(Q) $\quad q \in C(\overline{\Omega})$, $q \geq 0$, and $q(x_0) > 0$ for some $x_0 \in \Omega$.

We extend q to the whole of $\mathbb{R}^N$ by $q \equiv 0$.
Let $\xi \in \Omega$ be such that $q(\xi) > 0$, and
$\overline{B}_r(\xi) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^N : |x - \xi| \leq r\} \subset \Omega\,, \qquad 0 < T_0 < \infty.$
Then there exists some $T \in (0, T_0]$ such that the initial-boundary value problem (1) possesses a nontrivial nonnegative solution $u : \Omega \times (0, T) \to \mathbb{R}_+$ satisfying

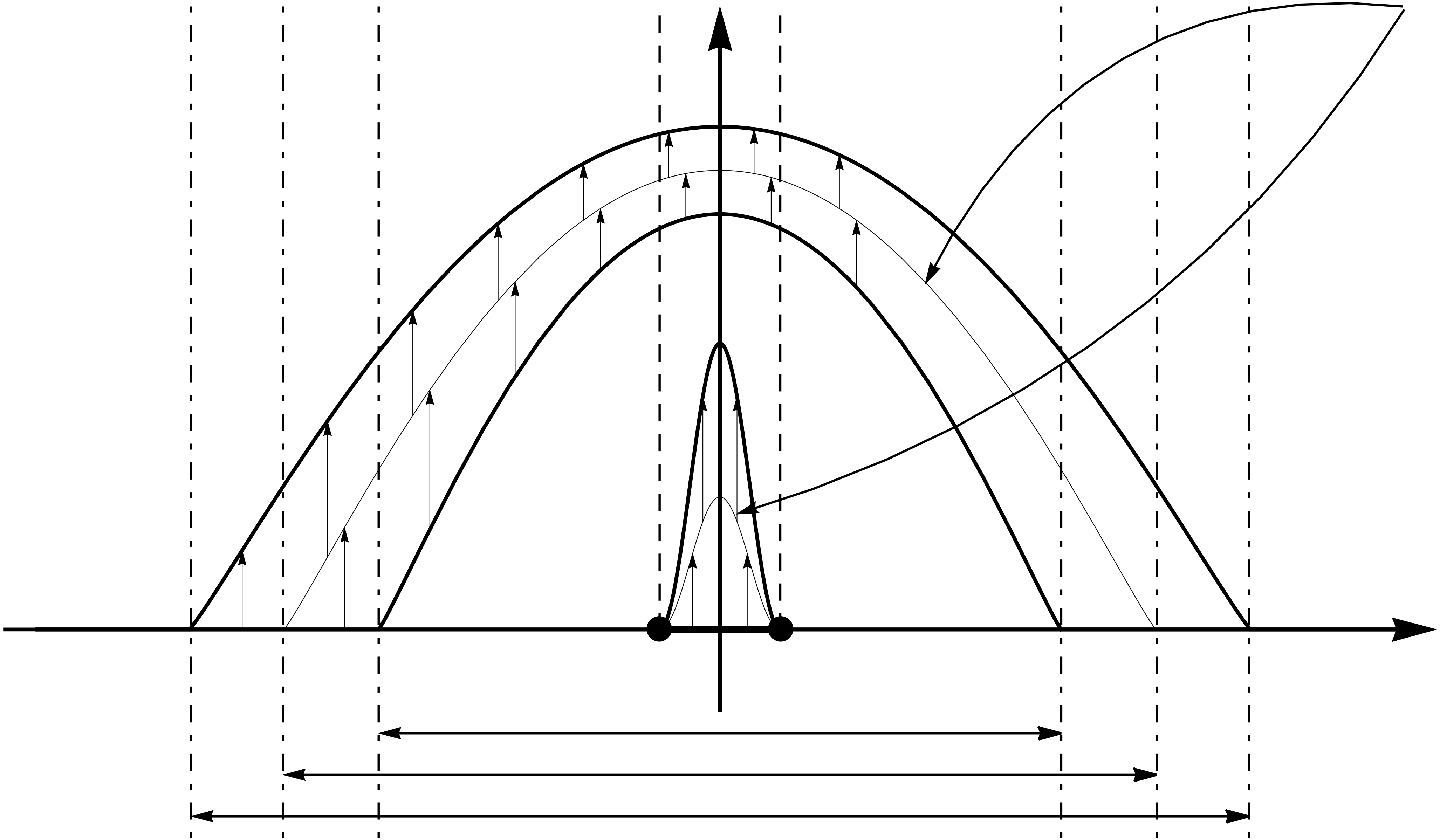
(i) $u(\xi, t) > 0$ for all $t \in (0, T)$;

(ii) $u(x, t) = 0$ for all $x \in \Omega \setminus B_r(\xi)$ and all $t \in (0, T)$.

In addition, if $\Omega = B_R(\xi)$ is a ball with radius R centered at ξ, $0 < r < R < \infty$, and q is radially symmetric about ξ, i.e., $q(x) \equiv q(|x - \xi|)$ for $x \in \Omega$, then the nontrivial solution u above can be constructed radially symmetric about ξ in the space variable $x \in \Omega$, i.e., $u(x, t) \equiv u(|x - \xi|, t)$.

Hence, u is a bump solution with at least one bump.

The **proofs** of both theorems above are based on constructing an ordered pair $\underline{u} \le \overline{u}$ of sub- and super-solutions to problem (1) with "bump properties" analogous to those of the desired solution.



Subsolution:

Assume that $2 < p < \infty$, $0 < \alpha < 1$, and (Q) are satisfied. Given a ball $B_R(x_0) \subset \Omega$ and a fixed number $T_0 \in (0,\infty)$, we define

$$\underline{u}(x,t) \stackrel{\text{def}}{=} \theta(t)\tilde{\varphi}_{1,R}(x)^{\beta} \quad \text{for } (x,t) \in \mathbb{R}^N \times [0,T_0]\,,$$

where $\beta > 1$, $\tilde{\varphi}_{1,R}$ is given by

$$\tilde{\varphi}_{1,R}(x) \stackrel{\text{def}}{=} \begin{cases} \varphi_{1,R}(x) & \text{for } x \in B_R(x_0)\,; \\ 0 & \text{for } x \in \mathbb{R}^N \setminus B_R(x_0)\,, \end{cases} \tag{6}$$

the natural zero extension of $\varphi_{1,R}$ from $B_R(x_0)$ to the whole of $\mathbb{R}^N$. Clearly, $\tilde{\varphi}_{1,R} \in W^{1,p}(\mathbb{R}^N)$.

$$\begin{gathered} -\Delta_p \varphi_{1,R} = \lambda_{1,R}\,\varphi_{1,R}^{p-1} \quad \text{in } B_R(x_0)\,; \\ \varphi_{1,R} = 0 \quad \text{on } \partial B_R(x_0)\,. \end{gathered} \tag{7}$$

$\varphi_{1,R} \in W_0^{1,p}(B_R(x_0))$ is normalized by $\varphi_{1,R}(x_0) = 1$; this normalization yields $0 < \varphi_{1,R}(x) \le 1$, $\forall x \in B_R(x_0)$.

Finally, $\theta : [0, T_0] \to \mathbb{R}_+$ is the nonnegative solution of the Cauchy problem

$$\text{(8)} \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t}(t) = \frac{q_0}{2}\,\theta(t)^{\alpha} \quad \text{for } t \in (0, T_0)\,; \quad \theta(0) = 0\,,$$

such that $0 < \theta(t) < \infty$ for every $t \in (0, T_0)$. Then $\underline{u} : \mathbb{R}^N \times (0, T_0) \to \mathbb{R}_+$ is a subsolution of problem (1) in a smaller domain $\Omega \times (0, \underline{T})$, i.e., for $x \in \Omega$ and $t \in (0, \underline{T})$ only, where $\underline{T} \in (0, T_0)$ is small enough.

On the Infinite Propagation Speed in Parabolic Problems with the p-Laplacian in a Domain for $p < 2$

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[1]Supervisor

Quasilinear parabolic problem:

$$
(1) \quad \begin{cases} \frac{\partial}{\partial t} b\,(u(x,t)) - \Delta_p u(x,t) &= f(x,t) \quad \text{for } (x,t) \in \Omega \times (0,T); \\ u(x,0) &= u_0(x) \quad \text{for } x \in \Omega; \\ u(x,t) &= 0 \quad \text{for } (x,t) \in \partial\Omega \times (0,T)\,; \end{cases}
$$

- $\Omega \subseteq \mathbb{R}^N$, $N > 1$,
- $0 < T < +\infty$,
- $1 < p < 2$ (fast diffusion),
- $\Delta_p \equiv \mathrm{div}\left(|\nabla u|^{p-2}\, \nabla u \right)$,
- $f : \Omega \times (0,T) \to \mathbb{R}_+$, continuous,
- $u_0 : \overline{\Omega} \to \mathbb{R}_+$, continuous.

- $b \colon \mathbb{R}_+ \to \mathbb{R}_+$,
- $b(0) = 0$,
- $b \in C^1(0,+\infty)$ with $b' > 0$ on $(0,+\infty)$,
- $\displaystyle \lim_{s \to 0+} \frac{s^{2-p}\, b'(s)}{|\log s|^{p-1}} = 0\,.$

Question

Let there exists $\xi \in \Omega$ such that $u(\xi, t_0) > 0$ for $t_0 > 0$. Is the solution $u(\cdot, t_0)$ positive everywhere in Ω?

Previous work, which supports the affirmative answer:

1988 Chen and Di Benedetto [1] for b being an identity and spatially local result,

1994 Ivanov [2] for spatially local result,

2009 Khin and Su [3] for $\Omega = \mathbb{R}^N$.

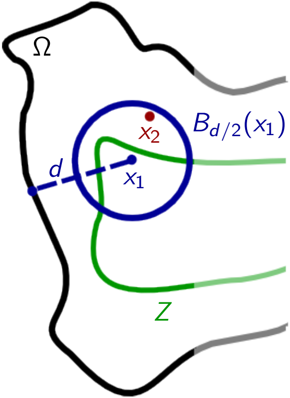
Modification of approach from [3] gives the affirmative answer also for the problem (1).

Theorem

Let $1 < p < 2$, $N \geq 1$ and assume that $b\colon \mathbb{R}_+ \to \mathbb{R}_+$ is as above. Finally, assume that $u\colon \overline{\Omega} \times [0, T) \to \mathbb{R}_+$ is a continuous, nonnegative, weak solution of (1). Then, for any fixed $t_0 \in (0, T)$, the solution $u(\cdot, t_0)$ is either positive everywhere on Ω or else identically zero on Ω.

Let us assume the following situation for fixed $t_0 \in (0, T)$:

- Denote $Z = \{x \in \Omega : u(x, t_0) = 0\}$,
- choose any $x_1 \in Z$ and denote $d = \operatorname{dist}(x_1, \delta\Omega)$,
- assume there exists $x_2 \notin Z$ such that $x_2 \in B_{d/2}(x_1)$.



Goal: Prove that $u(x_1, t_0) > 0$, which is in contradiction with the choice of $x_1 \in Z$.

Method: Construction of a nonnegative subsolution in the form of expanding spherical wave, which spreads from x_2 to x_1 (2009 Khin and Su [3]).